

# A NEW METHOD FOR SOURCE DETECTION, POWER ESTIMATION, AND LOCALIZATION IN LARGE SENSOR NETWORKS UNDER NOISE WITH UNKNOWN STATISTICS

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## ABSTRACT

Most statistical inference methods for array processing assume an array of size  $N$  fixed and a number of snapshots  $T$  large. In addition, many works are based on the assumption of a white noise model. These two assumptions are increasingly less realistic in modern systems where  $N$  and  $T$  are usually both large, and where the noise data can be correlated either across successive observations or across the sensor antennas. In this paper an approach to handle this kind of scenario is presented. New algorithms for source number estimation, power estimation, and localization by a sensor array under noise with unknown correlation model are proposed. The results fundamentally rely on recent advances in small rank perturbations of large dimensional random matrices.

**Index Terms**— Random matrix theory, correlated noise, source detection, power estimation, MUSIC algorithm.

## 1. INTRODUCTION

Consider a sensor network with  $N$  sensors observing  $T$  successive snapshots of  $K$  source signals. The received signals are impaired by temporally (or spatially) correlated noise, *i.e.*, there is a dependency between the noise data across successive observations (or across the sensors). Such scenarios are usually met in *e.g.*, radar systems. The objective of the sensor array is to determine the number of transmitting sources and to provide consistent estimators of their respective powers and angles of arrival using the  $T$  observations only.

In modern sensor networks, scenarios with large dimensional systems and fast dynamics where  $T$  is limited and is generally of the same order of magnitude as  $N$  are usually considered. Therefore, it is natural to assume the asymptotic regime denoted by  $T \rightarrow \infty$ , where  $T$  converges to infinity while  $N/T \rightarrow c > 0$ . The number of transmitting sources  $K$  is fixed as  $T \rightarrow \infty$ .

In this paper, apart from  $N$  and  $T$ , all parameters including  $K$  are unknown. In particular, the noise spatial or temporal correlations are unknown. The angle taken in this article to perform statistical inference on the signals is based on the spectral analysis of the empirical covariance matrix of the received signals.

The problem of detecting the number of sources has attracted a lot of attention. The historically used nonparametric estimators are the Akaike information criterion (AIC) [1] and the minimum description length metric (MDL) [2]. When  $N \rightarrow \infty$  and  $T \rightarrow \infty$ , an improved MDL estimator of  $K$  was recently proposed in [3]. Regarding power estimation, an  $(N, T)$ -consistent technique based on random matrix theory (RMT) was provided in [4]. As for angles of

arrival estimation, the MUSIC approaches are among the most popular techniques [5]. An improved RMT-based MUSIC algorithm was developed in [6].

The above detection/estimation techniques were elaborated for scenarios with white noise. When the noise is correlated and unknown, it is often assumed that the observer has access to an independent pure noise sequence. In this case, the noise covariance matrix is estimated and is used to whiten the received signal covariance matrix [7]. However, the existing detection/estimation techniques fail without the assumption that a pure noise sequence is available. In this paper it is assumed that such a sequence is not available. This is in particular more realistic in environments with fast dynamics.

Using recent results of RMT with finite rank perturbations [8], we elaborate new algorithms for the detection of the number of emitting sources and their power estimation provided the signal powers are large enough. Moreover, a MUSIC-based angle of arrival estimator is proposed which is an extension of the algorithm of [9] for the case of correlated noise. Our results are generalized in [10] where many system model constraints are relaxed, with proofs provided in [10] and [8].

The remainder of the article is organized as follows. In Section 2, we provide some background on RMT and introduce the system model. The detection, power estimation, and localization are presented in Section 3. Simulation results are given in Section 4.

## 2. BACKGROUND ON RMT AND SYSTEM MODEL

### 2.1. Background

We first introduce a result on the limiting eigenvalue distribution of a classical RMT model [11], [12].

**Theorem 1.** *Let  $V_T = W_T R_T^{1/2}$  be a matrix product where  $W_T$  has independent and identically distributed (i.i.d.) entries with zero mean, variance  $1/T$  and finite fourth order moment, and  $R_T$  is a deterministic nonnegative matrix with eigenvalue distribution  $\nu_T$  which converges to  $\nu$  as  $T \rightarrow \infty$ . Let  $\Sigma_T = \{\sigma_{i,T}^2\}_{i=1}^T$  be the set of eigenvalues of  $R_T$  with Hausdorff distance  $d(\Sigma_T, \text{supp}(\nu)) \xrightarrow{T \rightarrow \infty} 0$  where  $\text{supp}(\nu)$  is the support of  $\nu$ . Let  $\lambda_{1,T}, \dots, \lambda_{N,T}$  be the eigenvalues of  $V_T V_T^H$ . Then, as  $T \rightarrow \infty$ ,  $N/T \rightarrow c > 0$ , the eigenvalue distribution of  $V_T V_T^H$  converges to  $\mu$  whose Stieltjes transform  $\mathbf{m}(z)$ ,  $z \in \mathbb{C}^+$ , is given by the unique solution in  $\mathbb{C}_+$  of the equation*

$$\mathbf{m} = \left( -z + \int \frac{t}{1 + \mathbf{c}mt} \nu(dt) \right)^{-1}$$

where  $\mathbb{C}_+ = \{z \in \mathbb{C} : \Im z > 0\}$ . Moreover, for any interval

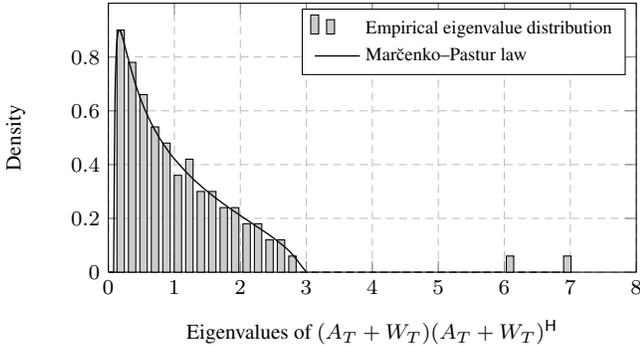
Work funded by the French Ile-de-France region, DIM/LSC fund, Digi-tec project DESIR

$[x_1, x_2] \subset \mathbb{R} - \text{supp}(\mu)$ ,

$\#\{i : \lambda_{i,T} \in [x_1, x_2]\} = 0$  almost surely (a.s.) for all large  $T$ .

When  $\nu = \delta_1$ , the limiting spectral measure of  $V_T V_T^H$  is the well known Marčenko–Pastur (MP) law [13] with parameter  $c$ .

Consider now the sum of matrices  $Y_T = A_T + V_T$  of size  $N \times T$  with  $V_T$  defined as in Theorem 1 and  $A_T$  with non-zero singular values  $\omega_1 \geq \dots \geq \omega_K > 0$  of rank  $K$  fixed. The matrix  $A_T$  can be viewed as a small rank perturbation of  $V_T$ . This model belongs to the literature of the so-called “spiked” models. Consider the empirical eigenvalue distribution of the matrix  $Y_T Y_T^H$  with eigenvalues  $\hat{\lambda}_1 \geq \dots \geq \hat{\lambda}_N$ . When  $\nu = \delta_1$ , the eigenvalue distribution of  $Y_T Y_T^H$  still converges to the MP law [14]. However, depending on  $\omega_i$  and  $c$ , we may observe up to  $K$  isolated eigenvalues on the right side of the support of  $\mu$  [15] (see Figure 1). In this paper we consider a model of the  $Y_T$  type with  $A_T$  modeling  $K$  source signals and  $V_T$  some colored thermal noise. The estimators presented in this paper are based on the asymptotic behavior of the isolated eigenvalues.



**Fig. 1.** Histogram of the empirical eigenvalues of  $(A_T + W_T)(A_T + W_T)^H$  and the Marčenko–Pastur law for  $N = 100$ ,  $T = 200$ ,  $A_T$  of rank 2 with  $\omega_1^2 = \omega_2^2 = 7$ .

## 2.2. System model

Consider  $K$  source signals received by an array of  $N$  sensors during  $T$  time slots. The received signal  $y_t \in \mathbb{C}^{N \times 1}$  at time  $t$  is given by

$$y_t = \sum_{k=1}^K \sqrt{p_k} a_T(\theta_k) s_{k,t} + v_t$$

where  $p_k$  is the power of source  $k$  ordered as  $p_1 \geq \dots \geq p_K$ ,  $\theta_k \in [-\pi/2, \pi/2]$  is its angle of arrival (different for each  $k$ ),  $a_T(\theta_k) \in \mathbb{C}^{N \times 1}$  is the steering vector defined in the classical uniform linear array model as

$$a_T(\theta_k) = \frac{1}{\sqrt{N}} \left[ 1, e^{-2i\pi d \sin \theta_k}, \dots, e^{-2i\pi d(N-1) \sin \theta_k} \right]^T$$

with  $d$  a positive constant. The signal transmitted by source  $k$  at time  $t$  is denoted by  $s_{k,t}$  and the noise vector by  $v_t$ . We can rewrite the input-output relationship by concatenating  $T$  successive signal realizations into the matrix

$$Y_T = H_T P^{1/2} S_T^H + V_T \quad (1)$$

where  $Y_T = [y_1, \dots, y_T]$ ,  $H_T = [a_T(\theta_1), \dots, a_T(\theta_K)]$ ,  $P = \text{diag}(p_1, \dots, p_K)$ ,  $S_T = T^{-1/2} [s_{t,k}^*]_{t,k=1}^{T,K}$  with  $s_{t,k}$  random i.i.d. with zero mean, unit variance, and finite eighth order moment, and  $V_T = [v_1, \dots, v_T]$ . We assume that the noise is temporally correlated, *i.e.*, the columns of  $V_T$  are not independent. Although this is not a necessary condition for the validity of the results of this paper (see [10] for a more general setting), we assume that the noise model is a causal stationary autoregressive moving average (ARMA) process. Each row of  $V_T$  is the output of a filter with transfer function of the form  $\mathbf{p}(z) = \sum_{l=0}^{\infty} \psi_l z^{-l}$  driven by a white Gaussian noise. Under these assumptions,  $V_T$  can be written as a product  $V_T = W_T R_T^{1/2}$  where  $W_T = T^{-1/2} [w_{n,t}]_{n,t=1}^{N,T}$  is a white noise matrix with  $w_{n,t}$  i.i.d. zero mean, unit variance, standard complex Gaussian random variables, and  $R_T$  is the Toeplitz nonnegative matrix

$$R_T = \begin{bmatrix} r_0 & r_1 & \dots & r_{T-1} \\ r_{-1} & \ddots & \ddots & \vdots \\ \vdots & \ddots & \ddots & r_1 \\ r_{1-T} & \dots & r_{-1} & r_0 \end{bmatrix}$$

with  $r_k \triangleq \sum_{l \geq 0} \psi_{l+k} \psi_l^*$  for  $k \in \mathbb{Z}$ . From [16, Lemma 6],  $\nu_T$ , the spectrum of  $R_T$ , converges to  $\nu$  whose support is a compact interval and all the eigenvalues of  $R_T$  are asymptotically located in  $\text{supp}(\nu)$ .

## 3. SOURCE DETECTION, POWER ESTIMATION, AND LOCALIZATION

From Theorem 1 and the discussion in Section 2.1, the spectrum of  $Y_T Y_T^H$  is constituted of one interval corresponding to the noise part and possibly of some isolated eigenvalues.

In the large dimensional regime with  $T \rightarrow \infty$ ,  $N/T \rightarrow c > 0$ , and  $K$  fixed, for this model, we precisely have, as a corollary of Theorem 1 and [16, Lemma 6], that the Stieltjes transform  $\mathbf{m}$  of the limiting eigenvalue distribution  $\mu$  of  $V_T V_T^H$  is the solution of the equation

$$\mathbf{m}(z) = \left( -z + \int_0^1 \frac{S(u)}{1 + c\mathbf{m}(z)S(u)} du \right)^{-1} \quad (2)$$

where, for  $z \in \mathbb{C}^+$ ,  $\mathbf{m}(z) \in \mathbb{C}^+$  and  $S(u) = |\mathbf{p}(e^{2\pi i u})|^2$ . The edge of the support of  $\mu$  is characterized in the following proposition [10].

**Proposition 1.** *Let  $\mu$  be the eigenvalue distribution function with the Stieltjes transform defined by (2) with support  $[a, b]$ . Then,*

$$b = -\frac{1}{\mathbf{m}_b} + \int_0^1 \frac{S(u)}{1 + c\mathbf{m}_b S(u)} du$$

where  $\mathbf{m}_b$  is the unique solution in  $(-c \max_u \{S(u)\}^{-1}, 0)$  of the equation in the variable  $\mathbf{m}$

$$\int_0^1 \left( \frac{\mathbf{m} S(u)}{1 + c\mathbf{m} S(u)} \right)^2 du = \frac{1}{c}.$$

Note that the function  $\mathbf{m}(z)$ ,  $z \in \mathbb{C}^+$ , admits an analytic continuation on  $(b, \infty)$  and  $\lim_{x \rightarrow b^+} \mathbf{m}(x) = \mathbf{m}_b$ .

### 3.1. Largest eigenvalues behavior

The behavior of the  $K$  largest eigenvalues of  $Y_T Y_T^H$  is known thanks to the recent results of [8] and given in the following proposition.

**Proposition 2.** Let  $\mathbf{m}$  be defined by (2) with  $\mu$  having support  $[a, b]$ . Let  $b$  and  $\mathbf{m}_b$  be defined as in Proposition 1. Define the function  $\mathbf{g}(x) = \mathbf{m}(x)(xc\mathbf{m}(x) + c - 1)$  decreasing from  $\mathbf{m}_b(cb\mathbf{m}_b + c - 1)$  to zero on  $(b, \infty)$ . Let  $q \in \mathbb{N}$  be the largest integer for which

$$p_q \mathbf{m}_b(cb\mathbf{m}_b + c - 1) > 1$$

and define  $p_{\text{lim}} \triangleq 1/\mathbf{m}_b(cb\mathbf{m}_b + c - 1)$ .

Let  $\hat{\lambda}_{1,T} \geq \dots \geq \hat{\lambda}_{N,T}$  be the eigenvalues of  $Y_T Y_T^H$ . If  $q = 0$ , then  $\hat{\lambda}_{1,T} \xrightarrow[T \rightarrow \infty]{a.s.} b$ . Otherwise, for  $k = 1, \dots, q$ , let  $\boldsymbol{\rho}_k$  be the unique solution in  $(b, \infty)$  of the equation  $p_k \mathbf{g}(x) = 1$ . Then,

$$\hat{\lambda}_{1,T} \xrightarrow[T \rightarrow \infty]{a.s.} \boldsymbol{\rho}_1, \dots, \hat{\lambda}_{q,T} \xrightarrow[T \rightarrow \infty]{a.s.} \boldsymbol{\rho}_q, \text{ and } \hat{\lambda}_{q+1,T} \xrightarrow[T \rightarrow \infty]{a.s.} b.$$

From this proposition, if  $q$  sources have their powers greater than  $p_{\text{lim}}$ , then the corresponding  $q$  eigenvalues will be found outside the support of  $\mu$ . Each of these eigenvalues  $\lambda_k$  converges to  $\boldsymbol{\rho}_k$  for  $k = 1, \dots, q$  and  $\boldsymbol{\rho}_k$  is a function of  $p_k$ . Therefore, the position of an isolated eigenvalue can be mapped to its corresponding source power, which can thus be estimated.

### 3.2. Source detection

Let  $L$  be an upperbound on the number of sources.

**Proposition 3.** Let  $q$  be defined as in Proposition 2. Let  $\varepsilon > 0$  and let  $\hat{q}_T^\varepsilon$  be the largest integer in  $\{0, \dots, L\}$  for which

$$\hat{q}_T^\varepsilon = \arg \max_{k \in \{0, \dots, L\}} \left\{ \frac{\hat{\lambda}_{k,T}}{\hat{\lambda}_{k+1,T}} > 1 + \varepsilon \right\}.$$

Then, for  $\varepsilon$  small enough,

$$\hat{q}_T^\varepsilon - q \xrightarrow[T \rightarrow \infty]{a.s.} 0.$$

From this result we have a consistent estimator of the source number if the source power  $p_K > p_{\text{lim}}$ , i.e., if  $q = K$ .

### 3.3. Power estimation

In order to estimate  $p_k$  for  $k \leq q$ , we use the equation  $p_k \mathbf{g}(\boldsymbol{\rho}_k) = 1$  from Proposition 2 where we replace  $\boldsymbol{\rho}_k$  and  $\mathbf{g}(x)$  by estimates based on  $\hat{\lambda}_{k,T}$ .

**Proposition 4.** In the setting of Proposition 3 with  $\hat{q}_T \triangleq \hat{q}_T^\varepsilon$  for some small  $\varepsilon$ , let  $\hat{g}_T(x) = \hat{m}_T(x)(xc\hat{m}_T(x) + c - 1)$  where  $\hat{m}_T(x)$  is given by

$$\hat{m}_T(x) = \frac{1}{N - \hat{q}_T} \sum_{n=\hat{q}_T+1}^N \frac{1}{\hat{\lambda}_{n,T} - x}.$$

For  $k = 1, \dots, \hat{q}_T$ , let  $\hat{p}_{k,T} = (\hat{g}_T(\hat{\lambda}_{k,T}))^{-1}$ .

Then,

$$\hat{p}_{k,T} - p_k \xrightarrow[T \rightarrow \infty]{a.s.} 0.$$

The variance of the proposed estimator can be shown to be of order  $1/T$  [10].

### 3.4. Localization

In this section we extend the method of [9] for the correlated noise model. Let  $q$  be defined as in Proposition 2. Let  $\Pi_{T,q}$  be an ortho-

nal projector on the column space of  $H_{T,q} = [a_T(\theta_1), \dots, a_T(\theta_q)]$ . As the signal subspace corresponding to the  $q$  sources is also spanned by the vectors  $a_T(\theta_1), \dots, a_T(\theta_q)$ ,  $\theta_1, \dots, \theta_q$  are solutions of the equation  $a_T(\theta) (I_N - \Pi_{T,q}) a_T(\theta)^H = 0$ . Defining  $\gamma_T(\theta) = a_T(\theta)^H \Pi_{T,q} a_T(\theta)$ , a localization function,  $\theta_1, \dots, \theta_q$  are the arguments of the local maxima of  $\gamma_T(\theta)$ . We can estimate  $\gamma_T(\theta)$  from the  $\hat{q}_T$  eigenvectors of  $Y_T Y_T^H$  corresponding to the  $\hat{q}_T$  largest eigenvalues as follows [10]

**Proposition 5.** Denote  $\hat{u}_{1,T}, \dots, \hat{u}_{\hat{q}_T,T}$  the eigenvectors of  $Y_T Y_T^H$  belonging respectively to  $\hat{\lambda}_{1,T}, \dots, \hat{\lambda}_{\hat{q}_T,T}$ . For  $\theta \in [-\pi/2, \pi/2]$ , let

$$\hat{\gamma}_T(\theta) = \sum_{k=1}^{\hat{q}_T} \zeta_T(\hat{\lambda}_{k,T}) a_T(\theta)^H \hat{u}_{k,T} \hat{u}_{k,T}^H a_T(\theta)$$

where  $\zeta_T(x) = \frac{(x\hat{m}_T(x)(c\hat{m}_T(x) - (1-c)\frac{1}{x}))'}{x\hat{m}_T(x)^2(c\hat{m}_T(x) - (1-c)\frac{1}{x})}$ .

Then,

$$\gamma_T(\theta) - \hat{\gamma}_T(\theta) \xrightarrow[T \rightarrow \infty]{a.s.} 0.$$

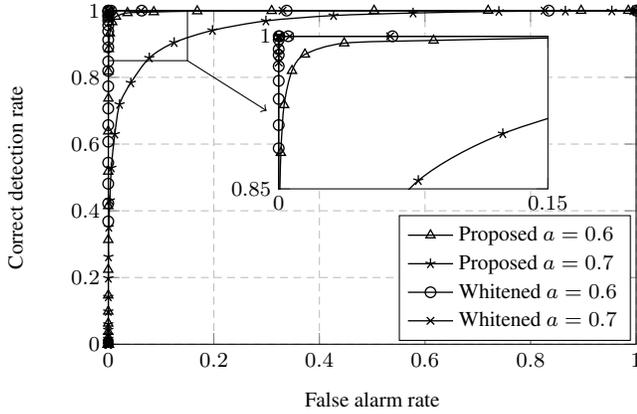
## 4. SIMULATION RESULTS

Simulation results are obtained for a single source located at  $\theta = 10^\circ$ . The signals  $s_{t,k}$  are drawn from a QPSK constellation. The noise is assumed to be an autoregressive (AR) process of order 1 and parameter  $a \in [0, 1)$ . The matrix  $R_T$  is then a Toeplitz matrix with coefficients  $[R_T]_{k,l} = a^{|k-l|}$ . In these simulations, we compare two cases. For the first case, we assume to have a perfect knowledge of  $R_T$ . Therefore, we first whiten the received covariance matrix before applying the proposed algorithm. This will give the theoretical performance upperbound. For the second case, we apply the proposed method directly on the received covariance matrix without knowledge of  $R_T$ .

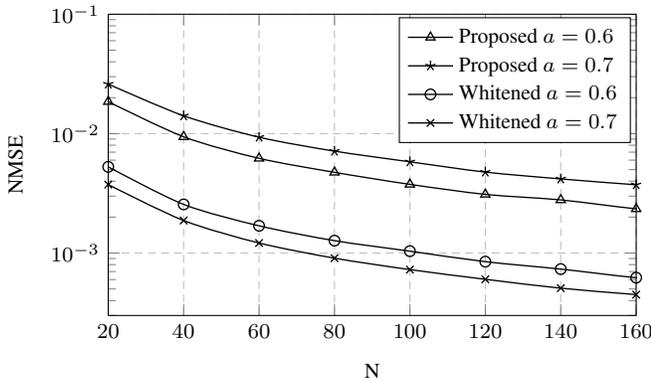
Figure 2 compares the receiver operation characteristics (ROC) of the detector proposed in Proposition 3 for different  $a$ . These curves are obtained by plotting the correct detection rate versus false alarm rate and are parameterized by the detection threshold. As expected, the whitened version significantly outperforms the proposed detector due to the perfect knowledge of  $R_T$ . Good detection performances are nonetheless observed for small values of  $a$ .

In Figure 3 normalized mean square errors (NMSE) of the power estimator given in Proposition 4 are depicted for different  $a$ . The proposed power NMSE is compared with the whitened version. The more correlated the noise, the bigger the gap between the NMSE of the proposed estimator and its whitened version at  $N$  fixed. It can be shown that the gap in dB is asymptotically given by  $10 \log \left( \int_0^1 \mathcal{S}(u)^{-1} du \int_0^1 \mathcal{S}(u) du \right)$  dB and is positive unless the noise is white (for which  $\mathcal{S}(u)$  is constant) [10]. We remark also that a good performance is achievable for a quite small  $N$ .

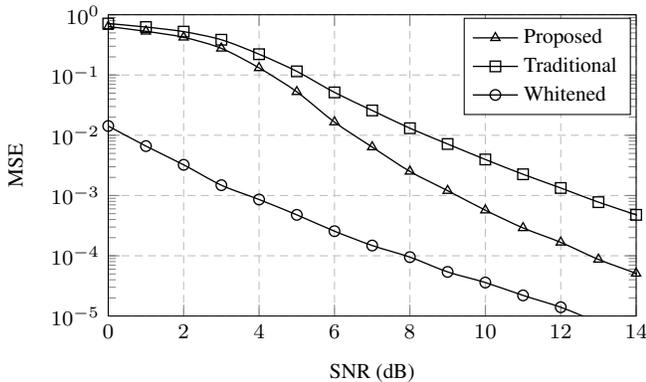
Figure 4 illustrates the mean square errors (MSE) of the estimated localization function of Proposition 5. They are compared with the traditional MUSIC algorithm with localization function  $\hat{\gamma}_{\text{trad},T}(\theta) = \sum_{k=1}^{\hat{q}_T} a_T(\theta)^H \hat{u}_{k,T} \hat{u}_{k,T}^H a_T(\theta)$ . The proposed localization algorithm outperforms the classical MUSIC algorithm, particularly at high SNR, by nearly 4 dB. Similar to the power estimation performance, the gap between the whitened version strongly depends on the correlation parameter  $a$ .



**Fig. 2.** ROC curves with  $K = 1$ ,  $\text{SNR} = 7$  dB,  $L = 5$ ,  $N = 20$ ,  $T = 40$ , and  $c = 0.5$ .



**Fig. 3.** NMSE of the estimated power versus  $N$  with  $K = 1$ ,  $\text{SNR} = 7$  dB,  $N/T = 0.5$ , and  $c = 0.5$ .



**Fig. 4.** MSE of the localization function versus SNR with  $K = 1$ ,  $N = 20$ ,  $T = 40$ ,  $c = 0.5$ , and  $a = 0.6$ .

## 5. CONCLUSIONS

In this paper new  $(N, T)$ -consistent estimators for the number of sources, their respective powers, and directions of arrival in the presence of noise with unknown correlation have been proposed. The simulations showed that the proposed algorithm performance strongly depends on the noise correlation parameter. However, when the correlation is not too high, a good performance is achievable for a reasonable system size. In addition, we have observed that the proposed angle of arrival scheme performs better than the traditional

MUSIC algorithm. All the proofs, as well as second order analysis, are available in [10] with more general conditions on the model.

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