Local failure localization in large sensor networks

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***Abstract*—The joint fluctuations of the extreme eigenvalues and eigenvectors of large sample covariance matrices of the spiked-model type are analyzed. This result is used to develop an original framework for the diagnosis of local failures in sensor networks, corroborated by simulations.**

I. Introduction

One of the elementary requests for fault diagnosis is the fast, reliable and computationally light identification of a system failure. In dynamical scenarios, those systems are made of fluctuating parameters whose evolutions are tracked by a noisy sensor measures, which become increasingly difficult to fast process in recent large networks. In this article, we concentrate on *off-line* diagnosis of *local* failures. We wish the diagnosis to be fast so we assume that the number n of successive sensor reports is not large compared to the size N of the network. We also assume the hypothetical failure scenarios known in advance. Calling $\mathcal{H}_0$ the hypothesis that the system is not in failure and $\mathcal{H}_k$, $1 \leq k \leq K$, the hypothesis that a failure of type k occurs, we need to: (i) decide whether the observation $\Sigma = [s_1, \ldots, s_n] \in \mathbb{C}^{N \times n}$ of n successive sensor reports suggests $\mathcal{H}_0$ or its complementary $\bar{\mathcal{H}}_0$ (the event union of the $\mathcal{H}_k$), and (ii) upon decision of $\bar{\mathcal{H}}_0$, decide which $\mathcal{H}_k$ is the most likely. Both problems are optimally solved by multi-hypothesis Neyman-Pearson tests with maximum likelihood performance given Σ but these tests are computationally intense for large systems.

The approach we follow is based on large dimensional random matrix theory and assumes $N, n \to \infty$ and $c_N = N/n \to c$, with $0 < c < \infty$. In this context, we develop asymptotic results on the extreme eigenvalues and associated eigenvector projections of a certain family of random matrices, in order to provide novel subspace methods for failure diagnosis. Our interest is on random matrices Σ modeled as $\Sigma = (I_N + P)^{\frac{1}{2}} X$ (called spiked model), where X is a left-unitarily invariant random matrix and P is a rank-r Hermitian matrix with $r \ll N$. Such matrix models have been largely studied recently, often in the special case where X is a *standard Gaussian matrix*, i.e. with independent $\mathcal{CN}(0, 1/n)$ entries [2], [3], [4], [1]. The main result of this article is an expression of the joint fluctuations of the extreme eigenvalues and associated eigenspace projections.

Notations: Uppercase characters stand for matrices, with $\|\cdot\|$ the spectral norm. Lowercase characters stand either for scalars or vectors, with $\|\cdot\|$ the Euclidean norm. The symbol $(\cdot)^*$ denotes complex transpose. We denote $\mathbb{C}^+ = \{z \in \mathbb{C},\ \Im(z) > 0\}$. The notations '$\xrightarrow{\text{a.s.}}$', '$\Rightarrow$', and '$\xrightarrow{\mathbb{P}}$' denote convergence almost sure, weak, and in probability, respectively.

II. Detection and localization of local failures

To motivate our study, we first introduce two examples of sensor network failure scenarios, which can be modeled as small rank perturbations of the identity matrix.

A. Node failure

Consider the model

$$y = H\theta + \sigma w \tag{1}$$

where $H \in \mathbb{C}^{N \times p}$ is deterministic, $\theta = [\theta(1), \ldots, \theta(p)]^\mathsf{T} \in \mathbb{C}^p$, $w \in \mathbb{C}^N$ have independent and identically distributed (i.i.d.) complex standard Gaussian entries, and $\sigma > 0$. We denote $y = [y(1), \ldots, y(N)]^\mathsf{T} \in \mathbb{C}^N$. In a sensor network composed of N nodes, y represents the observation through the channel H of the vector θ, constituted of independent Gaussian system parameters, impaired by white noise. Therefore, $\mathrm{E}[yy^*] = HH^* + \sigma^2 I_N \triangleq R$.

In case of failure of sensor k, $y(k)$ (k^{th} entry of y) will return inconsistent noisy outputs. Assuming this noise Gaussian with zero mean and variance σ_k^2 and denoting y' the modified observations of the network

$$y' = (I_N - e_k e_k^*)H\theta + \sigma_k e_k e_k^* \theta' + \sigma w$$

where θ' is distributed like θ, $e_k(k) = 1$ and $e_k(i) = 0$, $i \neq k$. We therefore now have that y' is Gaussian with zero mean and variance

$$\mathrm{E}[y'y'^*] = (I_N - e_k e_k^*)HH^*(I_N - e_k e_k^*) + \sigma_k^2 e_k e_k^* + \sigma^2 I_N.$$

Denoting $s = R^{-\frac{1}{2}} y'$, we have $\mathrm{E}[ss^*] = I_N + P_k$ with

$$P_k \triangleq R^{-\frac{1}{2}} e_k \left[(e_k^* HH^* e_k + \sigma_k^2) e_k^* R^{-\frac{1}{2}} - e_k^* HH^* R^{-\frac{1}{2}} \right] - R^{-\frac{1}{2}} HH^* e_k e_k^* R^{-\frac{1}{2}}. \tag{2}$$

Therefore, $\mathrm{E}[ss^*]$ is a perturbation of the identity matrix by P_k, whose image is included in $\mathrm{Span}(R^{-\frac{1}{2}} e_k, R^{-\frac{1}{2}} HH^* e_k)$ of dimension at most two.

B. Sudden parameter change

Consider again (1) and now assume that $\theta(k)$ experiences a sudden change in mean and variance, so y' is now

$$y' = H(I_p + \alpha_k e_k e_k^*)\theta + \mu_k H e_k + \sigma w$$

for some $\mu_k, \alpha_k \in \mathbb{R}$, and $e_k \in \mathbb{C}^p$ defined as above. The signal y' is now Gaussian with zero mean and variance

$$\mathrm{E}[y'y'^*] = H(I_p + [\mu_k^2 + (1 + \alpha_k)^2 - 1] e_k e_k^*)H^* + \sigma^2 I_N.$$

Denoting $R = HH^* + \sigma^2 I_N$ and taking $s = R^{-\frac{1}{2}}y'$, we then have $\mathrm{E}[ss^*] = I_N + P_k$ where $P_k = [\mu_k^2 + (1+\alpha_k)^2 - 1]R^{-\frac{1}{2}}He_ke_k^*H^*R^{-\frac{1}{2}}$ of unit rank.

The derivations above generalize naturally to sudden changes of multiple parameters.

C. Detection and localization

For models as above, assume a general scenario with K possible failure events, let $s_1, \ldots, s_n$ be n successive independent observations of the random variable s and denote $\Sigma \triangleq \frac{1}{\sqrt{n}}[s_1, \ldots, s_n] \in \mathbb{C}^{N\times n}$. We take s Gaussian with zero mean and covariance $(I_N + P_k)$ for a certain $k \in \{1, \ldots, K\}$, so we can write $\Sigma = (I_N + P_k)^{\frac{1}{2}}X$ where $X \in \mathbb{C}^{N\times n}$ is a given matrix with independent $\mathcal{CN}(0, 1/n)$ entries.

Since the failure information is carried by P_k, we derive in the following a likelihood test relying on the properties linking P_k to Σ, for large N, n. Precisely, we develop a two-step approach to: (i) decide on the occurrence of a failure and (ii) identify the failure, relying on the asymptotic statistics of the eigenstructure of $\Sigma\Sigma^*$.

III. Main results

A. Notations, assumptions and basic results

We first summarize our notations. We define $\Sigma = (I_N + P)^{\frac{1}{2}}X$ with $X \in \mathbb{C}^{N\times n}$ left-unitarily invariant, and P Hermitian of rank-r with spectral factorization $P = U\Omega U^*$, $\Omega = \mathrm{diag}\,(\omega_1 I_{j_1}, \ldots, \omega_t I_{j_t})$, $\omega_1 > \ldots > \omega_s > 0 > \omega_{s+1} > \ldots > \omega_t > -1$. Accordingly, $U = \left[U_1 \cdots U_t\right]$, $U_i \in \mathbb{C}^{N\times j_i}$. We denote $\hat{\lambda}_1 \geq \cdots \geq \hat{\lambda}_N$ the eigenvalues of $\Sigma\Sigma^*$. For $i \in \{1, \ldots, s\}$, let $\mathcal{K}(i) = j_1 + \cdots + j_{i-1}$, with $j_0 = 0$. For $i \in \{s{+}1, \ldots, t{+}1\}$, let $\mathcal{K}(i) = N - (j_i + \ldots + j_t)$. Also denote $\widehat{\Pi}_i$ the orthogonal projection matrix on the eigenspace of $\Sigma\Sigma^*$ associated with the eigenvalues $\{\hat{\lambda}_{\mathcal{K}(i)+\ell}\}_{\ell=1}^{j_i}$. Similarly, denote $\Pi_i = U_iU_i^*$ the orthogonal projection matrix on the eigenspace of P associated with ω_i. Finally, let $Q(z) \triangleq (XX^* - zI_N)^{-1}$ and $\alpha(z) = \frac{1}{N}\operatorname{tr} Q(z)$. We consider the asymptotic regime where $n, N \to \infty$ and $N/n \to c \in (0,1)$, denoted simply by $n \to \infty$.

We now state our basic assumptions:

- **A1** The probability law of X is invariant by left multiplication by a deterministic unitary matrix.
- **A2** For $z \in \mathbb{C}^+$, $\alpha(z) \xrightarrow{\text{a.s.}} m(z)$, the Stieltjes transform[1] of a measure π with support $[a, b] \subset (0, \infty)$.
- **A3** We have $\|XX^*\| \xrightarrow{\text{a.s.}} b$ and $(\|(XX^*)^{-1}\|)^{-1} \xrightarrow{\text{a.s.}} a$.

Assumption **A3** implies that **A2** holds also for $z \in \mathbb{C} \setminus [a, b]$.

The classical model satisfying **A1**-**A3** is for standard Gaussian X, i.e. with independent $\mathcal{C}N(0, 1/n)$ entries, as in the system models of Section II, for which the limiting distribution π is the Marčenko-Pastur distribution with Stieltjes transform

$$m(z) = \frac{1}{2zc}\left(1 - c - z + \sqrt{(1-c-z)^2 - 4zc}\right)$$

(the square root is such that $m(z) \in \mathbb{C}^+$ if $z \in \mathbb{C}^+$).

The unitary invariance of X implies the important lemma:

[1]We recall that the Stieltjes transform $m(z)$ of a real measure π is defined for z outside the support of π by $m(z) = \int \frac{1}{\lambda - z} d\pi(\lambda)$.

Lemma 1 ([6]): Assume **A1**. Let $u, v \in \mathbb{C}^N$ of unit norm, $\sigma(XX^*)$ the eigenvalue spectrum of XX^*. For $\varepsilon > 0$, $z \in \mathbb{C}\setminus[a-\varepsilon, b+\varepsilon]$, denote d_z the distance from z to $[a-\varepsilon, b+\varepsilon]$ and $A_N = \{\sigma(XX^*) \subset [a-\varepsilon, b+\varepsilon]\}$. Then, for $p > 0$,

$$\mathrm{E}\left\|1_{A_N} u^*(Q(z) - \alpha(z)I_N)v\right\|^p \leq \frac{K_p d_z^{-p}}{N^{p/2}}$$

where K_p depends on p only, and, for $z, z' \in \mathbb{C}\setminus[a-\varepsilon, b+\varepsilon]$,

$$\mathrm{E}\|1_{A_n} u^*(Q(z)Q(z') - \frac{1}{N}\operatorname{tr} Q(z)Q(z')I_N)v\|^p \leq \frac{K_p d_z^{-p} d_{z'}^{-p}}{N^{p/2}}.$$

We now introduce our main results and outlines of proofs.

B. First order behavior

1) Eigenvalues: We first write

$$\begin{aligned}
&\det(\Sigma\Sigma^* - xI_N)\\
&= \det(I_N + P)\det(XX^* - xI_N + x[I_N - (I_N + P)^{-1}])\\
&= \det(I_N + P)\det Q(x)\det(I_N + xP(I_N + P)^{-1}Q(x)).
\end{aligned}$$

If x is an eigenvalue of $\Sigma\Sigma^*$ but not of XX^*, it must cancel:

$$\begin{aligned}
&\det(I_N + xP(I_N + P)^{-1}Q(x))\\
&= \det(I_r + x\Omega U^*(I_N + U\Omega U^*)^{-1}Q(x)U).
\end{aligned}$$

Denote $\widehat{H}(z) = I_r + z\Omega(I_r + \Omega)^{-1}U^*Q(z)U$. Then under **A1**-**A3**, $\widehat{H}(z) \xrightarrow{\text{a.s.}} H(z) = I_r + zm(z)\Omega(I_r + \Omega)^{-1}$ for $z \in \mathbb{C} \setminus [a, b]$ (from Lemma 1). We therefore expect the solutions of $\det H(x) = 0$ outside $[a, b]$ to coincide with the limits of the isolated eigenvalues of $\Sigma\Sigma^*$, that is the solutions of

$$h(\rho) + (1 + \omega_k)\omega_k^{-1} = 0 \tag{3}$$

with $h(x) = xm(x)$, for some $k \in \{1, \ldots, p, q, \ldots, t\}$.

Theorem 1 ([7]): Assume **A1**-**A3**. Let p be zero or the maximum index such that $\omega_p > 0$ and $h(b^+) + (1 + \omega_p)\omega_p^{-1} < 0$, and q be $t + 1$ or the minimum index such that $\omega_q < 0$ and $h(a^-) + (1 + \omega_q)\omega_q^{-1} > 0$. For $i = 1, \ldots, p$, let ρ_i be the unique solution of (3) such that $\rho_i > b$. Then

$$\hat{\lambda}_{\mathcal{K}(i)+\ell} \xrightarrow{\text{a.s.}} \rho_i \text{ for } \ell = 1, \ldots, j_i \text{ and } \hat{\lambda}_{\mathcal{K}(p+1)+1} \xrightarrow{\text{a.s.}} b.$$

For $i = q, \ldots, t$, let ρ_i be the unique solution of (3) such that $\rho_i < a$. Then

$$\hat{\lambda}_{\mathcal{K}(i)+\ell} \xrightarrow{\text{a.s.}} \rho_i \text{ for } \ell = 1, \ldots, j_i \text{ and } \hat{\lambda}_{\mathcal{K}(q)} \xrightarrow{\text{a.s.}} a.$$

The variables $\omega_1, \ldots, \omega_p, \omega_q, \ldots, \omega_t$ satisfying the conditions of Theorem 1 are said to satisfy *the separation condition*. For X standard Gaussian, Theorem 1 entails:

Corollary 1: Consider the setting of Theorem 1. Assume additionally that X is standard Gaussian. Let p be zero or the maximum index for which $\omega_p > \sqrt{c}$ and q be $t + 1$ or the minimum index such that $\omega_q < -\sqrt{c}$. Then

$$\begin{aligned}
\hat{\lambda}_{\mathcal{K}(i)+\ell} &\xrightarrow{\text{a.s.}} \rho_i = 1 + \omega_i + c(1 + \omega_i)\omega_i^{-1}\\
\hat{\lambda}_{\mathcal{K}(p+1)+1} &\xrightarrow{\text{a.s.}} (1 + \sqrt{c})^2\\
\hat{\lambda}_{\mathcal{K}(q)} &\xrightarrow{\text{a.s.}} (1 - \sqrt{c})^2
\end{aligned}$$

for $i \in \{1, \ldots p, q, \ldots, t\}$ and $\ell = 1, \ldots, j_i$.

For failure detection purposes, upon observation of Σ, we may then test the null hypothesis $\Sigma = X$ (call it hypothesis

$\mathcal{H}_0$) against the hypothesis $\Sigma = (I_N + P)^{\frac{1}{2}}X$ (call it hypothesis $\bar{\mathcal{H}}_0$), depending on whether eigenvalues of $\Sigma\Sigma^*$ are found outside the support of the Marčenko-Pastur law. However, this information, if sufficient for failure detection purposes, is not good enough to perform failure localization. We need for this to consider eigenspace properties of P.

2) Projections on eigenspaces: Given $i \leq t$, we now assume that ω_i satisfies the separation condition. Given $b_1, b_2 \in \mathbb{C}^N$ with bounded norms, our purpose is to study the asymptotic behavior of $b_1^*\widehat{\Pi}_i b_2$. We shall show that this bilinear form is simply related with $b_1^*\Pi_i b_2$ in the asymptotic regime.

Our starting point is to express $b_1^*\widehat{\Pi}_i b_2$ as a Cauchy integral. Denoting $\mathcal{C}_i$ a positively oriented contour encompassing only the eigenvalues $\hat{\lambda}_{\mathcal{K}(i)+\ell}$ of $\Sigma\Sigma^*$ for $\ell = 1, \ldots, j_i$, we have after immediate calculus

$$\begin{aligned} b_1^*\widehat{\Pi}_i b_2 &= -\frac{1}{2\pi\imath}\oint_{\mathcal{C}_i} b_1^*(\Sigma\Sigma^* - zI_N)^{-1}b_2 \; dz \\ &= -\frac{1}{2\pi\imath}\oint_{\mathcal{C}_i} b_1^*(I_N + P)^{-\frac{1}{2}}Q(z)(I_N+P)^{-\frac{1}{2}}b_2 \; dz \\ &\quad + \frac{1}{2\pi\imath}\oint_{\mathcal{C}_i} \hat{a}_1(z)^*\widehat{H}(z)^{-1}\hat{a}_2(z) \; dz. \end{aligned}$$

where $\hat{a}_1(z)^* = zb_1^*(I_N + P)^{-\frac{1}{2}}Q(z)U$ and $\hat{a}_2(z) = \Omega(I_r + \Omega)^{-1}U^*Q(z)(I_N+P)^{-\frac{1}{2}}b_2$. By Assumption **A3** and Theorem 1, with probability one for all large n, the first term on the right hand side is zero, while the second is equal to

$$\frac{1}{2\pi\imath}\oint_{\gamma_i} \hat{a}_1(z)^*\widehat{H}(z)^{-1}\hat{a}_2(z)dz.$$

where γ_i is a deterministic positively oriented circle enclosing only ρ_i among the limits of the isolated eigenvalues specified by Theorem 1, therefore enclosing none of the ρ_j, $j \neq i$. Using Lemma 1 in conjunction with the analyticity properties of the integrand, one can show that $\hat{a}_1(z)^*\widehat{H}(z)^{-1}\hat{a}_2(z)$ converges uniformly to $a_1(z)^*H(z)^{-1}a_2(z)$ on γ_i almost surely, where $a_1(z)^* = zm(z)b_1^*(I_N + P)^{-\frac{1}{2}}U$ and $a_2(z) = m(z)\Omega(I_r + \Omega)^{-1}U^*(I_N + P)^{-\frac{1}{2}}b_2$. It results that $b_1^*\widehat{\Pi}_i b_2 - T_i \xrightarrow{\text{a.s.}} 0$, where

$$T_i \triangleq \frac{1}{2\pi\imath}\oint_{\gamma_i} a_1(z)^*H(z)^{-1}a_2(z)dz.$$

Details can be found in [6] in a similar situation. Let us find the expression of T_i. Noticing that $H(z)^{-1} = \sum_{\ell=1}^t \left[1 + zm(z)\omega_\ell(1+\omega_\ell)^{-1}\right]^{-1}\mathcal{I}_\ell$ where $\mathcal{I}_\ell = \mathrm{diag}(\mathbf{0}, \ldots, \mathbf{0}, I_{j_\ell}, \mathbf{0}, \ldots, \mathbf{0})$, we obtain

$$\begin{aligned} T_i &= \sum_{\ell=1}^t \frac{\omega_\ell}{(1+\omega_\ell)^2} b_1^*\Pi_\ell b_2 \frac{1}{2\pi\imath}\oint_{\gamma_i} \frac{zm^2(z)}{1 + zm(z)\frac{\omega_\ell}{1+\omega_\ell}}dz \\ &= \sum_{\ell=1}^t \frac{b_1^*\Pi_\ell b_2}{1+\omega_\ell}\frac{1}{2\pi\imath}\oint_{\gamma_i} \frac{zm^2(z)}{\frac{1+\omega_\ell}{\omega_\ell} + zm(z)}dz. \end{aligned}$$

Applying the residue theorem and observing from (3) that $(1+\omega_i)^{-1} = (1 + h(\rho_i))h(\rho_i)^{-1}$, we obtain the following limits.

Theorem 2: Assume **A1**-**A3**. Given $i \leq t$, assume that ω_i satisfies a separation condition. Let $b_1 \in \mathbb{C}^N$ and $b_2 \in \mathbb{C}^N$ be two sequences of increasing size deterministic vectors with bounded Euclidean norms. Then

$$b_1^*\widehat{\Pi}_i b_2 - \zeta_i b_1^*\Pi_i b_2 \xrightarrow{\text{a.s.}} 0$$

where $\zeta_i = m(\rho_i)(1 + h(\rho_i))(h'(\rho_i))^{-1}$.

Corollary 2: Let X be standard Gaussian, then Theorem 2 holds with $\zeta_i = (1 - c\omega_i^{-2})(1 + c\omega_i^{-1})^{-1}$.

Theorem 4 and Corollary 4 provide an interesting characterization of the eigenspaces of P through limiting projections in the large dimensional setting. In the context of local failure in large sensor networks, it is therefore possible to detect and diagnose one or multiple failures by comparing eigenspace projection patterns associated with each failure type. To this end though, not only first order limits but also second order behaviour need be characterized precisely.

C. Second order behavior

Before studying the fluctuations of $\hat{\lambda}_{\mathcal{K}(i)+\ell}$, $\ell = 1, \ldots, j_i$, when ω_i satisfies the separation property, we first remind for later use the fluctuations of $\hat{\lambda}_{\mathcal{K}(i)+\ell}$ when ω_i does *not* satisfy the separation property, and when X is a standard Gaussian matrix. For this, we have the following theorem [4].

Theorem 3: Let X be standard Gaussian. If $0 < \omega_i < \sqrt{c}$,

$$N^{\frac{2}{3}}(1+\sqrt{c})^{-\frac{4}{3}}c^{-\frac{1}{2}}(\hat{\lambda}_{\mathcal{K}(i)+\ell} - (1+\sqrt{c})^2) \Rightarrow T_2$$

and, if $-\sqrt{c} < \omega_i < 0$,

$$-N^{\frac{2}{3}}(1-\sqrt{c})^{-\frac{4}{3}}c^{-\frac{1}{2}}(\hat{\lambda}_{\mathcal{K}(i)+\ell} - (1-\sqrt{c})^2)) \Rightarrow T_2$$

for $\ell = 1, \ldots, j_i$, as $n \to \infty$, where T_2 is the complex Tracy-Widom distribution function [8].

The tools used to derive Theorem 3 are different from those exploited here and are not discussed. For failure diagnosis purposes, Theorem 3 will be used to declare a failure prior to locate it. To locate the fault, second order statistics of both eigenvalue and eigenspace projections when the separation property arises are needed.

In contrast to above, we now assume that ω_i satisfies the separation property. We first need additional assumptions:

A4 For $z \in \mathbb{C} \setminus [a, b]$, $\sqrt{N}\,(\alpha(z) - m(z)) \xrightarrow{\mathbb{P}} 0$.

This assumption is satisfied by most models of practical importance in our context, provided $\sqrt{N}(c_N - c) \to 0$. For simplicity, we also assume:

A5 Each ω_i, $1 \leq i \leq t$, satisfies the separation condition.

The main result of this section is the following theorem.

Theorem 4: Assume **A1**-**A5**. For $i = 1, \ldots, t$, we denote

$$\begin{aligned} L_{i,n} &= \sqrt{N}\left[\hat{\lambda}_{\mathcal{K}(i)+1} - \rho_i, \ldots, \hat{\lambda}_{\mathcal{K}(i)+j_i} - \rho_i\right]^{\mathsf{T}} \\ V_{i,n} &= \sqrt{N}U_i^*\left(\widehat{\Pi}_i - \zeta_i I_N\right)U_i. \end{aligned}$$

For $\rho \in \mathbb{R} \setminus [a, b]$, let

$$D(\rho) \triangleq \begin{bmatrix} \frac{h(\rho)(1+h(\rho))h''(\rho)}{h'(\rho)^3} & -\frac{h(\rho)(1+h(\rho))}{h'(\rho)^2} \\ -\frac{\rho}{h'(\rho)} & 0 \end{bmatrix}$$

$$R(\rho) \triangleq \begin{bmatrix} m'(\rho) - m(\rho)^2 & m''(\rho)/2 - m(\rho)m'(\rho) \\ m''(\rho)/2 - m(\rho)m'(\rho) & m^{(3)}(\rho)/6 - m'(\rho)^2 \end{bmatrix}$$

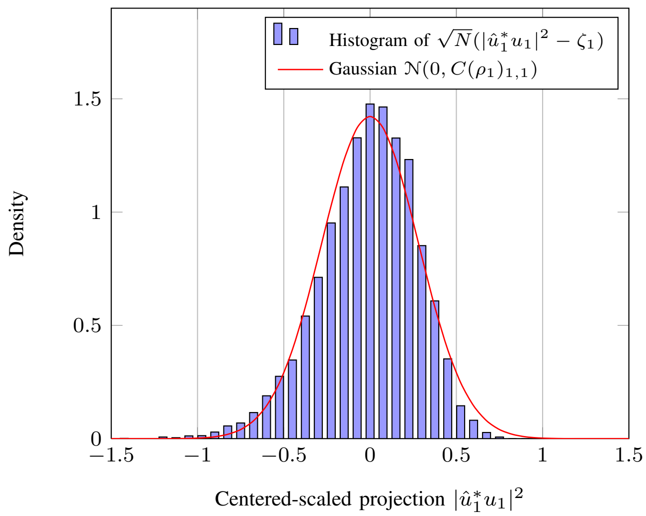

Fig. 1. Empirical and theoretical distribution of the fluctuations of $\hat{u}_1$ with $r = 1$, X has i.i.d. zero mean variance $1/n$ entries, $N/n = 1/8$, $N = 64$ and $\omega_1 = 1$.

where $m^{(3)}$ is the third derivative of m. Consider the matrices

$$\begin{bmatrix} G_i \\ K_i \end{bmatrix} \triangleq \left((D(\rho_i)R(\rho_i)D(\rho_i)^*)^{1/2} \otimes I_{j_i} \right) \begin{bmatrix} M_{1,i} \\ M_{2,i} \end{bmatrix}$$

where $M_{1,1}, M_{2,1}, \ldots, M_{1,t}, M_{2,t}$ are independent GUE matrices with $M_{1,i}, M_{2,i} \in \mathbb{C}^{j_i \times j_i}$. Let L_i be the $\mathbb{R}^{j_i}$-valued vector of eigenvalues of K_i arranged in decreasing order. Then

$$((V_{i,n}, L_{i,n}))_{i=1}^t \Rightarrow ((G_i, L_i))_{i=1}^t .$$

When the multiplicities of the eigenvalues of P are all equal to one, we immediately have the following corollary:

Corollary 3: If $j_i = 1$ for all i, Theorem 4 becomes

$$((V_{i,n}, L_{i,n}))_{i=1,\ldots,r} \Rightarrow \mathcal{N}(0, R)$$

with $R = \mathrm{diag}(D(\rho_1)R(\rho_1)D(\rho_1)^*, \ldots, D(\rho_r)R(\rho_r)D(\rho_r)^*)$.

In the standard Gaussian case, we have in particular:

Corollary 4: If X is standard Gaussian, Corollary 3 holds with $D(\rho_i)R(\rho_i)D(\rho_i)^* = C(\rho_i)$, with

$$C(\rho_i) \triangleq \begin{bmatrix} \frac{c^2(1+\omega_i)^2}{(c+\omega_i)^2(\omega_i^2-c)}\left(c\frac{(1+\omega_i)^2}{(c+\omega_i)^2}+1\right) & \frac{(1+\omega_i)^3c^2}{(\omega_i+c)^2\omega_i} \\ \frac{(1+\omega_i)^3c^2}{(\omega_i+c)^2\omega_i} & \frac{c(1+\omega_i)^2(\omega_i^2-c)}{\omega_i^2} \end{bmatrix}.$$

In Figure 1, the histogram of a simulation of 10 000 realizations of the projection $V_{1,n} = \sqrt{N}(|\hat{u}_1^* u_1|^2 - \zeta_1)$, with $u_1 = U_1 \in \mathbb{C}^N$, $\hat{u}_1\hat{u}_1^* = \hat{\Pi}_1$ of unit rank, and X standard Gaussian, is depicted against the asymptotic Gaussian law derived in Corollary 4, for $c = 1/8$, $r = 1$, $N/n = 1/8$, $N = 64$ and ω_1 successively equal to 1.

We now sketch the proof of Corollary 3. Theorem 4 is more difficult to treat using the contour integration method and requires other approaches mimicking [9]. Using residue calculus, we first find

$$L_{i,n} - \left[-\frac{\rho_i}{h'(\rho_i)} u_i^*(m(\rho_i) - Q(\rho_i))u_i \right] \xrightarrow{\text{a.s.}} 0$$

$$V_{i,n} - \left[\frac{h(\rho_i)(1+h(\rho_i))h''(\rho_i)}{h'(\rho_i)^3} u_i^*(m(\rho_i) - Q(\rho_i))u_i - \frac{h(\rho_i)(1+h(\rho_i))}{h'(\rho_i)^2} u_i^*(m'(\rho_i) - Q'(\rho_i))u_i \right] \xrightarrow{\text{a.s.}} 0$$

It then suffices to study the joint fluctuations of $u_i^*(m(\rho_i) - Q(\rho_i))u_i$ and $u_i^*(m'(\rho_i) - Q'(\rho_i))u_i$. For this, we have the following lemma, proved in [10] (and generalized to $j_i > 1$):

Lemma 2: Assume **A1**-**A4**, and $j_i = 1$. Let $f_1, \ldots, f_t$ and $g_1, \ldots, g_t$ be real functions analytical on a neighborhood of $[a, b]$. Let S_n be the t-uple of random matrices

$$S_n = \left(\sqrt{N} \begin{bmatrix} u_i^* f_i(XX^*)u_i - \left(\int f_i(\lambda)d\pi(\lambda)\right) \\ u_i^* g_i(XX^*)u_i - \left(\int g_i(\lambda)d\pi(\lambda)\right) \end{bmatrix} \right)_{i=1}^t .$$

For $i = 1, \ldots, t$, define the covariance matrices

$$R_i = \int \left(\begin{bmatrix} f_i - \int f_i d\pi \\ g_i - \int g_i d\pi \end{bmatrix} \begin{bmatrix} (f_i - \int f_i d\pi) & (g_i - \int g_i d\pi) \end{bmatrix} \right) d\pi.$$

Then

$$S_n \Rightarrow \mathcal{N}(0, \mathrm{diag}(R_1, \ldots, R_t)).$$

Applying this lemma with with $f_i(\lambda) = (\lambda - \rho_i)^{-1}$ and $g_i(\lambda) = (\lambda - \rho_i)^{-2}$, R_i takes the value $R(\rho_i)$ provided in the statement of Theorem 4. An immediate application of the delta method then gives the final result.

IV. Application

Recall that we assume a number K of failure scenarios indexed by $1 \leq k \leq K$. Scenario k is modelled by $\Sigma = (I_N + P_k)^{\frac{1}{2}} X$ with $P_k = \sum_{i=1}^{t_k} \omega_{k,i} U_{k,i} U_{k,i}^*$ of rank $r_k = \sum_{i=1}^{t_k} j_{k,i}$, where $U_{k,i} \in \mathbb{C}^{N \times j_{k,i}}$ and $\omega_{k,1} > \ldots > \omega_{k,s_k} > 0 > \omega_{k,s_k+1} > \ldots > \omega_{k,t_k}$, and X is standard Gaussian. We also call p_k the smallest index i such that $\omega_{k,i} > \sqrt{c}$ (or zero), and q_k the largest index i such that $\omega_{k,i} < -\sqrt{c}$ (or $t_k + 1$).

A. Detection algorithm

The detection phase relies on Theorem 3. We decide here between $\mathcal{H}_0$ and $\bar{\mathcal{H}}_0$. For simplicity, assume that all P_k only have non-negative eigenvalues. From Theorem 1, the largest eigenvalue $\hat{\lambda}_1$ of $\Sigma\Sigma^*$ tends to $(1 + \sqrt{c})^2$ under $\mathcal{H}_0$, while $\hat{\lambda}_1$ is larger than $(1 + \sqrt{c})^2$ under $\bar{\mathcal{H}}_0$ *if* $\omega_{k,1}$ exceeds $\sqrt{c}$. We assume that $\omega_{k,1} > \sqrt{c}$ is verified for all k. That is, we assume that $c_N \triangleq N/n < c_+$ where $c_+ \triangleq \inf \{c \mid \omega_{k,1} > \sqrt{c},\ 1 \leq k \leq K\}$. This condition allows for a theoretically almost sure error detection, as $N, n \to \infty$. The test consists in rejecting $\mathcal{H}_0$ if its posterior probability is sufficiently low. That is, for a given acceptable *false alarm rate* η, the statistical test is defined as

$$\hat{\lambda}_1' \underset{\bar{\mathcal{H}}_0}{\overset{\mathcal{H}_0}{\lessgtr}} (T_2)^{-1}(1 - \eta) \qquad (4)$$

where $\hat{\lambda}_1' \triangleq N^{\frac{2}{3}}(1 + \sqrt{c_N})^{-\frac{4}{3}} c_N^{-\frac{1}{2}} (\hat{\lambda}_1 - (1 + \sqrt{c_N})^2)$. This generalizes naturally to scenarios where eigenvalues of P_k may be of arbitrary sign, see [10].

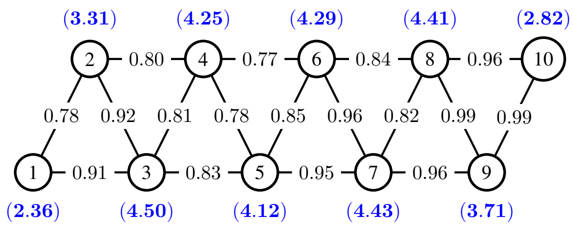

Fig. 2. Network of $N = 10$ sensors. The correlation $\mathrm{E}[y(i)^* y(j)]$, $i \neq j$, is read on link (i, j), and $\mathrm{E}[|y(i)|^2]$ is shown in parentheses.

B. Localization algorithm

We now wish to diagnose the identified failure. Similar to the previous sections, we denote $\rho_{k,i} = 1 + \omega_{k,i} + c(1 + \omega_{k,i})\omega_{k,i}^{-1}$ and $\zeta_{k,i} = (1 - c\omega_{k,i}^{-2})(1 + c\omega_{k,i}^{-1})^{-1}$, we define the mapping $\mathcal{K}_k$ to be such that $\mathcal{K}_k(i) = j_{k,1} + \ldots + j_{k,i-1}$ if $1 \leq i \leq s_k$, and $\mathcal{K}_k(i) = N - (j_{k,1} + \ldots + j_{k,t_k})$ if $s_k + 1 \leq i \leq t_k$. Finally, we denote $\widehat{\Pi}_{k,i}$ any projector on the subspace generated by the eigenvalues $\hat{\lambda}_{\mathcal{K}_k(i)+1}, \ldots, \hat{\lambda}_{\mathcal{K}_k(i)+j_{k,i}}$.

Since different P_k's have in general distinct eigenspaces, we propose the following subspace localization test, which decides on the hypothesis $\mathcal{H}_{k^\star}$ for which $k^\star$ is given by

$$k^\star = \arg\max_{k \in S} \mathbb{P}\left(\left(V_{i,n}^k, L_{i,n}^k\right)_{i \in \mathcal{L}(p_k, q_k)} \right) \tag{5}$$

with $\mathcal{L}(p_k, q_k) = \{1, \ldots, p_k, q_k, \ldots, r_k\}$, S the set of indexes k such that $\mathcal{L}(p_k, q_k)$ is non-empty, and where

$$L_{i,n}^k \triangleq \sqrt{N} \left[\hat{\lambda}_{\mathcal{K}_k(i)+1} - \rho_{k,i}, \ldots, \hat{\lambda}_{\mathcal{K}_k(i)+j_{k,i}} - \rho_{k,i} \right]^{\mathsf{T}}$$
$$V_{i,n}^k \triangleq \sqrt{N} U_{k,i}^* \left(\widehat{\Pi}_{k,i} - \zeta_{k,i} I_{j_{k,i}} \right) U_{k,i}.$$

We need here to specify the indexation $i \in \mathcal{L}(p_k, q_k)$ since we do not assume **A5** for any k. From Theorem 4, this probability can be approximated for large n. When the $\omega_{k,i}$ all have multiplicity one, Corollary 4 gives the following estimator

$$\hat{k} = \arg\max_{k \in S} \prod_{l \in \mathcal{L}(p_k, p_k)} f\left((V_{i,n}^k, L_{i,n}^k); C(\rho_{k,i}) \right) \tag{6}$$

where $f(x; \Omega)$, $x \in \mathbb{C}^m$, $\Omega \in \mathbb{C}^{m \times m}$, is the m-variate real normal density of zero mean and covariance C at x, and $C(\rho_{k,i})$ is defined as in Corollary 4, with ω_i replaced by $\omega_{k,i}$.

C. Simulations

Our application example relates to the sensor network model $y = H\theta + \sigma w$ of Section II-A for $N = 10$ nodes, $p = N$, and $\sigma^2 = -20$ dB. This is depicted in Figure 2, where the entries of $HH^* + \sigma^2 I_N$ are presented. We also take $\sigma_k^2 = \sum_{i=1}^N (HH^*)_{ki}$. In this context, it appears that, for all k, $\omega_{k,1} \ll |\omega_{k,2}|$. We therefore only consider the largest eigenvalue of $\Sigma\Sigma^*$ to perform the failure diagnosis. The theoretical threshold for $c_N = N/n$ (if N, n were large) is 0.8 with the worst-case failure being on node 10. We carry out $100\,000$ Monte Carlo simulations of node 10 failures for n varying from 8 to 140 and under false alarm rates varying from 10^{-2} to 10^{-4}. This is depicted in Figure 3, where we observe that detection becomes possible for $n = 8$. For not too large n, while detection rates increase, we observe that localization capabilities are still unsatisfying, which is mainly due to the inappropriate fit of the large dimensional model with $N = 10$. This is corrected for larger n.

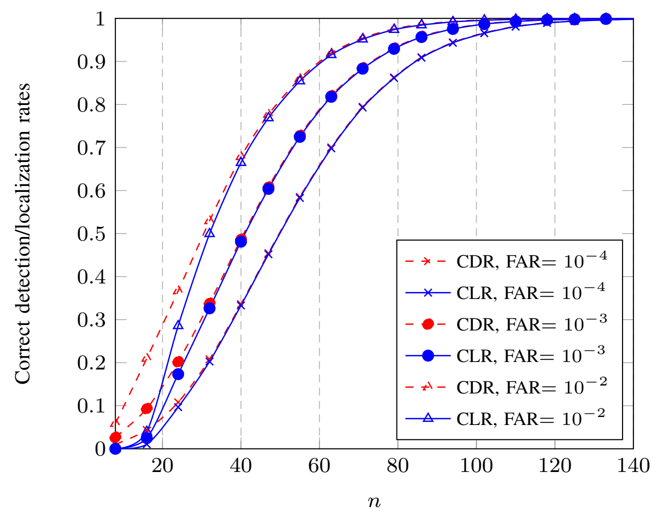

Fig. 3. Correct detection (CDR) and localization (CLR) rates for different false alarm rates (FAR) and different n.

V. Conclusion

The joint fluctuations of the extreme eigenvalues and corresponding eigenspace projections of spiked random matrices are evaluated and used to develop novel failure diagnosis algorithms in large sensor networks. The minimal number of observations required is evaluated and simulations are provided that suggest the algorithms allow for high failure diagnosis performance even in small size networks.

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