Large System Analysis of Zero-Forcing Precoding in MISO Broadcast Channels with Limited Feedback

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***Abstract*—In this paper we analyze the sum-rate of zero-forcing (ZF) precoding in MISO broadcast channels with limited feedback, transmit correlation and path loss. Our analysis assumes that the number of transmit antennas M and the number of users K are large, while their ratio remains bounded. By applying recent results from random matrix theory we derive a deterministic equivalent of the SINR and compute the sum-rate maximizing number of users as well as the limiting sum-rate for high SNR, as a function of the channel uncertainties and the channel correlation pattern. Simulations show that theoretical and numerical results match well, even for small system dimensions.**

I. Introduction

The capacity achieving precoding strategy of the Gaussian MIMO broadcast channel based on the non-linear dirty-paper coding (DPC) technique [1]. But so far no efficient practical algorithm implementing the optimal DPC scheme has been proposed. Therefore, low complexity linear precoding strategies have gained a lot of attention since they achieve a large portion of the channel capacity at moderate complexity.

A classical linear interference mitigating scheme is zero-forcing (ZF) precoding which has first been analyzed in the context of MIMO broadcast channels in [2].

In this contribution we consider a system where both the number of transmit antennas M and the number of users K are large but their ratio $\beta(M) \triangleq M/K$ is bounded. We extend the models of [3]–[5] by considering imperfect channel state information at the transmitter (CSIT), transmit correlation as well as different path losses of the users. With the aid of recent tools from random matrix theory (RMT), we derive a *deterministic equivalent* of the signal-to-interference plus noise ratio (SINR) of ZF precoding which is *independent* of the individual channel realizations. From the deterministic equivalent of the SINR we determine the sum-rate maximizing number of users.

Notation: In the following boldface lower-case and upper-case characters denote vectors and matrices, respectively. The operators $(\cdot)^{\mathsf{H}}$, $\mathrm{tr}(\cdot)$ and $\mathrm{Tr}(\cdot)$ denote conjugate transpose, trace and normalized matrix trace, respectively. The expectation is $E[\cdot]$ and $\mathrm{diag}(\mathbf{x})$ is a diagonal matrix with vector $\mathbf{x}$ on the main diagonal. The $N\times N$ identity matrix is $\mathbf{I}_N$.

II. Mathematical Preliminaries

Definition 1 (Deterministic Equivalent): Let $\{\mathbf{X}_M : M = 1,2,\dots\}$ be a set of complex random matrices $\mathbf{X}_M$ of size $M\times M$. For some functional f we define a *deterministic equivalent* $m^\circ_{\mathbf{X}_M}$ of $m_{\mathbf{X}_M} \triangleq f(\mathbf{X}_M)$ as any series $m^\circ_{\mathbf{X}_1}, m^\circ_{\mathbf{X}_2},\dots$ such that

$$m_{\mathbf{X}_M} - m^\circ_{\mathbf{X}_M} \overset{M\to\infty}{\longrightarrow} 0 \tag{1}$$

almost surely.
In present work we are interested in deterministic equivalents of expressions of the form

$$m_{\mathbf{B}_K,\mathbf{Q}_K}(z) = \mathrm{Tr}\mathbf{Q}_K\left(\mathbf{B}_K - z\mathbf{I}_K\right)^{-1} \tag{2}$$

where $\mathbf{Q}_K \in \mathbb{C}^{K\times K}$ is invertible and $\mathbf{B}_K \in \mathbb{C}^{K\times K}$ is a random matrix of the type

$$\mathbf{B}_K = \mathbf{T}_K^{1/2}\mathbf{X}_K\mathbf{R}_K\mathbf{X}_K^{\mathsf{H}}\mathbf{T}_K^{1/2} \tag{3}$$

where $\mathbf{R}_K \in \mathbb{C}^{M\times M}$ is positive Hermitian, $\mathbf{T}_K \in \mathbb{C}^{K\times K}$ is diagonal and $\mathbf{X}_K \in \mathbb{C}^{K\times M}$ is random with independent and identically distributed (i.i.d.) entries of zero mean and variance $1/K$. In the course of the derivations, we will require the following result,

Theorem 1: Under the above model for $\mathbf{B}_K$ where $\mathbf{T}_K$ and $\mathbf{R}_K$, $\mathbf{Q}_K$ have uniformly bounded spectral norm (w.r.t. M), as (K,M) grow large with ratio β such that $0<\beta = M/K \le \infty$, for $z\in\mathbb{C}$, $\mathrm{Im}(z)>0$,

$$m_{\mathbf{B}_K,\mathbf{Q}_K}(z) - m^\circ_{\mathbf{B}_K,\mathbf{Q}_K}(z) \overset{M\to\infty}{\longrightarrow} 0 \tag{4}$$

almost surely, where $m^\circ_{\mathbf{B}_K,\mathbf{Q}_K}(z)$ is defined as

$$m^\circ_{\mathbf{B}_K,\mathbf{Q}_K}(z) = \mathrm{Tr}\mathbf{Q}_K\left(c(z)\mathbf{T}_K - z\mathbf{I}_K\right)^{-1} \tag{5}$$

$$\text{with} \quad c(z) = \mathrm{Tr}\mathbf{R}_K\left(\mathbf{I}_M + \frac{1}{\beta}e(z)\mathbf{R}_K\right)^{-1} \tag{6}$$

and $e(z)$ is the unique solution, with $\mathrm{Im}(e(z))>0$, of

$$e(z) = \mathrm{Tr}\mathbf{T}_K\left(c(z)\mathbf{T}_K - z\mathbf{I}_K\right)^{-1} \tag{7}$$

The proof of Theorem 1 is a straightforward extension of the proof provided in [6]. Note that $m_{\mathbf{B}_K}(z) \triangleq m_{\mathbf{B}_K,\mathbf{I}_M}(z)$ is the Stieltjes transform [7] of the eigenvalue distribution of $\mathbf{B}_K$.

III. System Model

Consider the MISO broadcast channel composed of one central transmitter equipped with M antennas and of K single-antenna receivers. Assume $M > K$ and narrow-band communication. Denoting y_k the signal received by user k, the concatenated received signal vector $\mathbf{y} = [y_1, \dots, y_K]^{\mathsf{T}} \in \mathbb{C}^K$ at a given time instant reads

$$\mathbf{y} = \sqrt{M}\mathbf{H}\mathbf{x} + \mathbf{n} \tag{8}$$

with transmit vector $\mathbf{x} \in \mathbb{C}^M$, channel matrix $\mathbf{H} \in \mathbb{C}^{K\times M}$ and noise vector $\mathbf{n} \sim \mathcal{CN}(0, \sigma^2\mathbf{I}_K)$. The transmit vector $\mathbf{x}$ is obtained by linear precoding $\mathbf{x} = \mathbf{G}\mathbf{s}$, where $\mathbf{s} \sim \mathcal{CN}(0, \mathbf{I}_K)$ is the symbol vector and $\mathbf{G} = [\mathbf{g}_1, \dots, \mathbf{g}_K] \in \mathbb{C}^{M\times K}$ is the precoding matrix. The total transmit power is $P > 0$, hence

$$\mathrm{tr}(E[\mathbf{x}\mathbf{x}^{\mathsf{H}}]) = \mathrm{tr}(\mathbf{G}\mathbf{G}^{\mathsf{H}}) \le P. \tag{9}$$

In this paper we consider ZF precoding i.e.

$$\mathbf{G} = \frac{\xi}{\sqrt{M}}\left(\tilde{\mathbf{H}}^{\mathsf{H}}\tilde{\mathbf{H}}\right)^{-1}\tilde{\mathbf{H}}^{\mathsf{H}} \tag{10}$$

where $\tilde{\mathbf{H}}$ is the estimated channel matrix available at the transmitter and the scaling factor ξ is set to fulfill the power constraint (9). From (9) we then obtain

$$\xi^2 = \frac{P}{\frac{1}{M}\mathrm{tr}\left(\tilde{\mathbf{H}}^{\mathsf{H}}\tilde{\mathbf{H}}\right)^{-1}} = \frac{P}{m_{\tilde{\mathbf{H}}^{\mathsf{H}}\tilde{\mathbf{H}}}(0)} \triangleq \frac{P}{\bar{\Psi}}. \tag{11}$$

The received symbol y_k of user k is given by

$$y_k = \xi\mathbf{h}_k^{\mathsf{H}}(\tilde{\mathbf{H}}^{\mathsf{H}}\tilde{\mathbf{H}})^{-1}\tilde{\mathbf{h}}_k s_k + \xi\sum_{i=1, i\neq k}^{K}\mathbf{h}_k^{\mathsf{H}}(\tilde{\mathbf{H}}^{\mathsf{H}}\tilde{\mathbf{H}})^{-1}\tilde{\mathbf{h}}_i s_i + n_k$$

where $\mathbf{h}_k^{\mathsf{H}}$ and $\tilde{\mathbf{h}}_k^{\mathsf{H}}$ denote the kth *row* of $\mathbf{H}$ and $\tilde{\mathbf{H}}$, respectively. The SINR $\gamma_{k,\mathrm{zf}}$ of user k can be written in the form

$$\gamma_{k,\mathrm{zf}} = \frac{|\mathbf{h}_k^{\mathsf{H}}(\tilde{\mathbf{H}}^{\mathsf{H}}\tilde{\mathbf{H}})^{-1}\tilde{\mathbf{h}}_k|^2}{\mathbf{h}_k^{\mathsf{H}}(\tilde{\mathbf{H}}^{\mathsf{H}}\tilde{\mathbf{H}})^{-1}\tilde{\mathbf{H}}_k^{\mathsf{H}}\tilde{\mathbf{H}}_k(\tilde{\mathbf{H}}^{\mathsf{H}}\tilde{\mathbf{H}})^{-1}\mathbf{h}_k + \frac{1}{\rho}\bar{\Psi}} \tag{12}$$

where $\tilde{\mathbf{H}}_k^{\mathsf{H}} = [\tilde{\mathbf{h}}_1, \dots, \tilde{\mathbf{h}}_{k-1}, \tilde{\mathbf{h}}_{k+1}, \dots, \tilde{\mathbf{h}}_K] \in \mathbb{C}^{M\times(K-1)}$ and $\rho = P/\sigma^2$ denotes the SNR. The sum-rate R_{sum} is given by

$$R_{\mathrm{sum}} = \sum_{k=1}^{K}\log(1 + \gamma_{k,\mathrm{zf}}) \quad [\mathrm{nats/s/Hz}]. \tag{13}$$

Under the assumption of a rich scattering environment the correlated channel can be modeled as [8]–[10]

$$\mathbf{H} = \mathbf{L}^{1/2}\mathbf{H}_w\mathbf{\Theta}_{\mathrm{T}}^{1/2} \tag{14}$$

where $\mathbf{H}_w \in \mathbb{C}^{K\times M}$ has Gaussian i.i.d. zero-mean entries of variance $1/M$, $\mathbf{\Theta}_{\mathrm{T}} \in \mathbb{C}^{M\times M}$ is the nonnegative definite correlation matrix at the transmitter with eigenvalues $(\lambda_1, \dots, \lambda_M)$ and $\mathbf{L} = \mathrm{diag}([l_1, \dots, l_K])$ is a diagonal matrix containing the user's channel gain, i.e. the inverse path losses. We assume $\|\mathbf{\Theta}_{\mathrm{T}}\|$ to be uniformly bounded from above with respect to M, i.e. adding more transmit antennas does not significantly increase the correlation between them.

Moreover, we suppose that only $\tilde{\mathbf{H}}$, an imperfect estimate of the true channel matrix $\mathbf{H}$, is available at the transmitter. The channel-gain matrix $\mathbf{L}$ as well as the transmit correlation $\mathbf{\Theta}_{\mathrm{T}}$ are assumed to be slowly varying compared to the channel $\tilde{\mathbf{H}}_w$ and are perfectly known. We therefore model $\tilde{\mathbf{H}}$ as

$$\tilde{\mathbf{H}} = \mathbf{L}^{1/2}\tilde{\mathbf{H}}_w\mathbf{\Theta}_{\mathrm{T}}^{1/2} \tag{15}$$

$$\text{with} \quad \tilde{\mathbf{H}}_w = \sqrt{1-\tau^2}\mathbf{H}_w + \tau\mathbf{Q} \tag{16}$$

where $\mathbf{Q} \in \mathbb{C}^{K\times M}$ has Gaussian i.i.d. zero-mean entries of variance $1/M$ and $\tau \in [0,1]$ reflects the amount of distortion in the CSIT. Furthermore, we suppose that $\mathbf{H}_w$ and $\mathbf{Q}$ are mutually independent as well as independent of the symbol vector $\mathbf{s}$ and noise $\mathbf{n}$. A similar model for imperfect CSIT has been used in [11]–[13].

IV. Deterministic Equivalent of the SINR

In the following we derive a *deterministic equivalent* $\gamma_{k,\mathrm{zf}}^{\circ}$ of the SINR $\gamma_{k,\mathrm{zf}}$ of user k for ZF precoding, i.e. $\gamma_{k,\mathrm{zf}}^{\circ}$ is such that, almost surely,

$$\gamma_{k,\mathrm{zf}} - \gamma_{k,\mathrm{zf}}^{\circ} \xrightarrow{M\to\infty} 0. \tag{17}$$

That is, $\gamma_{k,\mathrm{zf}}^{\circ}$ is an approximation of $\gamma_{k,\mathrm{zf}}$ independent of the particular realizations of $\mathbf{H}_w$ and $\mathbf{Q}$. In [14] we derived a deterministic equivalent for regularized ZF (RZF) precoding. The same techniques as for RZF, cannot be applied for ZF, since by removing a row of $\tilde{\mathbf{H}}$ the matrix $(\tilde{\mathbf{H}}^{\mathsf{H}}\tilde{\mathbf{H}})^{-1}$ becomes singular. Therefore we will adopt a different strategy. Starting from the SINR expression of RZF for large (K, M), we will let the regularization term α go to zero.

The SINR $\gamma_{k,\mathrm{rzf}}$ of user k of RZF for large (K, M) is given by [14]

$$\gamma_{k,\mathrm{rzf}} = \frac{l_k^2(1-\tau^2)m_{\mathbf{A}}^2}{l_k\Upsilon\left[1 + l_k\tau^2(l_k m_{\mathbf{A}} + 2)m_{\mathbf{A}}\right] + \frac{\Psi(\alpha)}{\rho}(1 + l_k m_{\mathbf{A}})^2} \tag{18}$$

where $\mathbf{A} = \tilde{\mathbf{H}}_w^{\mathsf{H}}\mathbf{L}\tilde{\mathbf{H}}_w + \alpha\mathbf{\Theta}_{\mathrm{T}}^{-1}$ and

$$m_{\mathbf{A}} \triangleq m_{\mathbf{A}}(0) = \mathrm{Tr}\mathbf{A}^{-1} \tag{19}$$

$$\Upsilon = m_{\mathbf{A}} - \alpha\mathrm{Tr}\mathbf{\Theta}_{\mathrm{T}}^{-1}\mathbf{A}^{-2} \tag{20}$$

$$\Psi(\alpha) = \mathrm{Tr}\tilde{\mathbf{H}}^{\mathsf{H}}\tilde{\mathbf{H}}\left(\tilde{\mathbf{H}}^{\mathsf{H}}\tilde{\mathbf{H}} + \alpha\mathbf{I}_M\right)^{-2} \tag{21}$$

For $\beta > 1$, we will derive

$$\gamma_{k,\mathrm{zf}} = \lim_{\alpha\to 0}\gamma_{k,\mathrm{rzf}} \tag{22}$$

and subsequently find a deterministic equivalent $\gamma_{k,\mathrm{zf}}^{\circ}$ for $\gamma_{k,\mathrm{zf}}$. Equations (19) and (20) depend on α through $\mathbf{A}$. Therefore we will expand $\mathbf{A}^{-1}$ around $\alpha = 0$ and then take the limit $\alpha \to 0$. We can rewrite $m_{\mathbf{A}} = \mathrm{Tr}\mathbf{A}^{-1}$ as

$$\begin{aligned} m_{\mathbf{A}} &\overset{(a)}{=} \frac{1}{\alpha}\mathrm{Tr}\mathbf{\Theta}_{\mathrm{T}} - \frac{1}{\alpha M}\mathrm{tr}\tilde{\mathbf{H}}\mathbf{\Theta}_{\mathrm{T}}\tilde{\mathbf{H}}^{\mathsf{H}}(\tilde{\mathbf{H}}\tilde{\mathbf{H}}^{\mathsf{H}} + \alpha\mathbf{I}_K)^{-1} \\ &\overset{(b)}{\approx} \frac{1}{\alpha}\mathrm{Tr}\mathbf{\Theta}_{\mathrm{T}} - \frac{1}{\alpha M}\mathrm{tr}\tilde{\mathbf{H}}\mathbf{\Theta}_{\mathrm{T}}\tilde{\mathbf{H}}^{\mathsf{H}}\left[(\tilde{\mathbf{H}}\tilde{\mathbf{H}}^{\mathsf{H}})^{-1} - \alpha(\tilde{\mathbf{H}}\tilde{\mathbf{H}}^{\mathsf{H}})^{-2}\right] \end{aligned} \tag{23}$$

where (a) follows from the matrix inversion lemma (MIL) and in (b) we rewrite the inverse in terms of a Taylor series of order 2 around the point $\alpha = 0$. Taking the trace of $\alpha\mathbf{\Theta}_{\mathrm{T}}^{-1}\mathbf{A}^{-2}$, we obtain

$$\alpha\mathrm{Tr}\mathbf{\Theta}_{\mathrm{T}}^{-1}\mathbf{A}^{-2} \approx \frac{1}{\alpha}\mathrm{Tr}\mathbf{\Theta}_{\mathrm{T}} - \frac{1}{\alpha M}\mathrm{tr}\tilde{\mathbf{H}}\mathbf{\Theta}_{\mathrm{T}}\tilde{\mathbf{H}}^{\mathsf{H}}(\tilde{\mathbf{H}}\tilde{\mathbf{H}}^{\mathsf{H}})^{-1} + \frac{\alpha}{M}\mathrm{tr}\tilde{\mathbf{H}}\mathbf{\Theta}_{\mathrm{T}}\tilde{\mathbf{H}}^{\mathsf{H}}(\tilde{\mathbf{H}}\tilde{\mathbf{H}}^{\mathsf{H}})^{-3}. \quad (24)$$

Substituting (23) and (24) into (20), we obtain

$$\bar{\Upsilon} = \lim_{\alpha\to 0}\Upsilon = \frac{1}{M}\mathrm{tr}\tilde{\mathbf{H}}\mathbf{\Theta}_{\mathrm{T}}\tilde{\mathbf{H}}^{\mathsf{H}}\left(\tilde{\mathbf{H}}\tilde{\mathbf{H}}^{\mathsf{H}}\right)^{-2}. \quad (25)$$

Replacing $m_{\mathbf{A}}$, Υ and $\Psi(\alpha)$ in (18) with (23), (25) and $\bar{\Psi} = \Psi(0)$, respectively, we have

$$\gamma_{k,\mathrm{zf}} = \lim_{\alpha\to 0}\gamma_{k,\mathrm{rzf}} = \frac{1-\tau^2}{l_k\tau^2\bar{\Upsilon} + \frac{\bar{\Psi}}{\rho}}. \quad (26)$$

Now we derive a deterministic equivalent $\bar{\Psi}^{\circ}$ and $\bar{\Upsilon}^{\circ}$ for $\bar{\Psi}$ and $\bar{\Upsilon}$, respectively.

From Theorem 1 we find $\bar{\Psi}^{\circ}$, s.t. $\bar{\Psi} - \bar{\Psi}^{\circ} \xrightarrow{M\to\infty} 0$ almost surely, as

$$\bar{\Psi}^{\circ} = m^{\circ}_{\tilde{\mathbf{H}}\tilde{\mathbf{H}}^{\mathsf{H}}}(0) = \frac{1}{\beta c}\mathrm{Tr}\mathbf{L}^{-1} \quad (27)$$

where c is the unique solution of

$$c = \mathrm{Tr}\mathbf{\Theta}_{\mathrm{T}}\left(\mathbf{I}_M + \frac{1}{c\beta}\mathbf{\Theta}_{\mathrm{T}}\right)^{-1}. \quad (28)$$

In order to derive a deterministic equivalent $\bar{\Upsilon}^{\circ}$ for $\bar{\Upsilon}$ we need the following result

Lemma 1: [15] Let $\mathbf{A}$ be a deterministic $N \times N$ complex matrix with uniformly bounded spectral radius for all N. Let $\mathbf{x} \in \mathbb{C}^N$ have i.i.d. complex entries of zero mean, variance $1/N$ and finite eighth moment, then

$$\mathbf{x}^{\mathsf{H}}\mathbf{A}\mathbf{x} - \frac{1}{N}\mathrm{tr}\mathbf{A} \xrightarrow{N\to\infty} 0 \quad (29)$$

almost surely.

Since $\tilde{\mathbf{H}}_w$ is Gaussian and hence unitary invariant, $\mathbf{\Theta}_{\mathrm{T}}$ in (25) can be assumed diagonal. Denoting $\mathbf{C} = \tilde{\mathbf{H}}\tilde{\mathbf{H}}^{\mathsf{H}}$ and $\mathbf{C}_k = \tilde{\mathbf{H}}_k\tilde{\mathbf{H}}_k^{\mathsf{H}}$ and applying the MIL [16, Lemma 2.2] twice, Equation (25) takes the form

$$\bar{\Upsilon} = \frac{1}{M}\sum_{i=1}^{M}\lambda_i^2\frac{\mathbf{h}_{wk}^{\mathsf{H}}\mathbf{L}^{1/2}\mathbf{C}_k^{-2}\mathbf{L}^{1/2}\mathbf{h}_{wk}}{(1+\lambda_i\mathbf{h}_{wk}^{\mathsf{H}}\mathbf{L}^{1/2}\mathbf{C}_k\mathbf{L}^{1/2}\mathbf{h}_{wk})^2} \quad (30)$$

where $\mathbf{h}_{wk}$ is the kth column of $\mathbf{H}_w$. Applying Lemma 1 together with fact that $\mathrm{Tr}\mathbf{C}_k - \mathrm{Tr}\mathbf{C} \xrightarrow{M\to\infty} 0$ [14], we obtain

$$\bar{\Upsilon} = \frac{1}{\beta}\mathrm{Tr}\mathbf{L}\mathbf{C}^{-2}\frac{1}{M}\sum_{i=1}^{M}\frac{\lambda_i^2}{(1+\lambda_i\frac{1}{\beta}\mathrm{Tr}\mathbf{L}\mathbf{C}^{-1})^2}. \quad (31)$$

To determine the deterministic equivalent expression $m_{\mathbf{C},\mathbf{L}}(0)^{\circ}$ for $m_{\mathbf{C},\mathbf{L}}(0) = \mathrm{Tr}\mathbf{L}\mathbf{C}^{-1}$ we can directly apply Theorem 1. For $\mathrm{Tr}\mathbf{L}\mathbf{C}^{-2}$ we have

$$\mathrm{Tr}\mathbf{L}\mathbf{C}^{-2} = m_{\mathbf{C}^2,\mathbf{L}}(z) = \left.\frac{\partial m_{\mathbf{C}^2,\mathbf{L}}(z)}{\partial z}\right|_{z=0} = m'_{\mathbf{C},\mathbf{L}}(0). \quad (32)$$

The derivative of $m_{\mathbf{C},\mathbf{L}}(0)^{\circ}$ is a deterministic equivalent of $m'_{\mathbf{C},\mathbf{L}}(0)$, so that applied to (31), we have a deterministic equivalent $\bar{\Upsilon}^{\circ}$, s.t. $\bar{\Upsilon} - \bar{\Upsilon}^{\circ} \xrightarrow{M\to\infty} 0$ almost surely, that verifies

$$\bar{\Upsilon}^{\circ} = \frac{\frac{c_2}{c^2}}{\beta - \frac{c_2}{c^2}}\mathrm{Tr}\mathbf{L}^{-1} \quad (33)$$

$$\text{with} \quad c_2 = \mathrm{Tr}\mathbf{\Theta}_{\mathrm{T}}^2\left(\mathbf{I}_M + \frac{1}{c\beta}\mathbf{\Theta}_{\mathrm{T}}\right)^{-2}. \quad (34)$$

Note that $c_2/c^2 \geq 1$ with equality if $\mathbf{\Theta}_{\mathrm{T}} = \mathbf{I}_M$. Finally $\gamma^{\circ}_{k,\mathrm{zf}}$ is given by

$$\gamma^{\circ}_{k,\mathrm{zf}} = \frac{1-\tau^2}{l_k\tau^2\bar{\Upsilon}^{\circ} + \frac{\bar{\Psi}^{\circ}}{\rho}}. \quad (35)$$

Note that the computation of (35) involves the evaluation of only *one* fixed-point equation, given by (28).

From (35) it follows that the sum rate $R^{\mathrm{zf}}_{\mathrm{sum}}$ saturates at asymptotically high SNR

$$R^{\mathrm{lim}}_{\mathrm{sum}} \triangleq \lim_{\rho\to\infty}\sum_{k=1}^{K}\log(1+\gamma^{\circ}_{k,\mathrm{zf}}) = \sum_{k=1}^{K}\log\left(1+\frac{\beta(1-\tau^2)}{\tau^2 l_k\bar{\Upsilon}^{\circ}}\right). \quad (36)$$

V. Asymptotically Optimal Number Of Users K

In this work we assume a limited feedback channel of constant rate, leading to a *fixed* distortion τ^2 of the CSIT. We now deal with the problem of finding the optimal number of users $K^{\star}$ that maximizes the *deterministic equivalent* of the sum-rate for fixed $\mathbf{\Theta}_{\mathrm{T}}$, $\mathbf{L}$, ρ and τ^2. This is equivalent to optimizing β for a fixed M. For large (K, M) we define $\beta^{\star}$ as

$$\beta^{\star} = \arg\max_{\beta>1}\frac{1}{\beta}\int\log\left(1+\gamma^{\circ}_{k,\mathrm{zf}}(\mathbf{L},\beta)\right)p(\mathbf{L})d\mathbf{L}. \quad (37)$$

where we assumed that the channel gains l_k are distributed according to $p(\mathbf{L})$. In particular we will suppose that the users are distributed uniformly on a ring centered at the base-station.

By setting the derivative of the RHS of (37) w.r.t. β to zero we obtain the implicit equation

$$\beta\int\frac{\frac{\partial\gamma^{\circ}_{k,\mathrm{zf}}}{\partial\beta}p(\mathbf{L})d\mathbf{L}}{1+\gamma^{\circ}_{k,\mathrm{zf}}(\mathbf{L},\beta)} = \int\log\left(1+\gamma^{\circ}_{k,\mathrm{zf}}(\mathbf{L},\beta)\right)p(\mathbf{L})d\mathbf{L}. \quad (38)$$

Therefore $\beta^{\star}$ is the solution to (38).

In the special case of $\mathbf{\Theta}_{\mathrm{T}} = \mathbf{I}_M$ and $\mathbf{L} = \mathbf{I}_K$ the solution to (38) has an explicit form. Equation (35) takes the form

$$\gamma^{\circ}_{k,\mathrm{zf}} = \frac{1-\tau^2}{\tau^2+\frac{1}{\rho}}(\beta-1). \quad (39)$$

and we can write the solution to (37) explicitly. For equation (38) we obtain

$$\frac{a\beta}{1+a(\beta-1)} = \log\left(1+a(\beta-1)\right) \quad (40)$$

where $a = \frac{1-\tau^2}{\tau^2+\frac{1}{\rho}}$. Denoting

$$w(\beta) = \frac{a-1}{a(\beta-1)+1} \quad \text{and} \quad x = \frac{a-1}{e}, \quad (41)$$

we can rewrite (40) as

$$w(\beta)e^{w(\beta)} = x. \quad (42)$$

Notice that $w(\beta) = \mathcal{W}(x)$, where $\mathcal{W}(x)$ is the Lambert W-function defined as $z = \mathcal{W}(z)e^{\mathcal{W}(z)}$, $z \in \mathbb{C}$. Therefore, by solving $w(\beta) = \mathcal{W}(x)$ we have

$$\beta^\star = \left(1 - \frac{1}{a}\right)\left(1 + \frac{1}{\mathcal{W}(x)}\right). \quad (43)$$

For $\tau \in [0,1]$, $\beta > 1$ we have $w \geq -1$ and $x \in [-e^{-1}, \infty)$. In this case $\mathcal{W}(x)$ is a single-valued function. If $\tau = 0$, we obtain the results in [5] and [17]. Note that only rational values of β are meaningful in practice.

If the transmit antennas are spaced sufficiently apart the major loss in sum-rate is due to path loss, cf. Figure 1. Therefore, it is of interest to characterize the sum-rate gap R_Δ between a user distribution $p(\mathbf{L})$ and equally distant users $\mathbf{L} = \mathbf{I}_K$. For a *fixed* β and with $\tau^2 = 0$, we have (35)

$$R_\Delta = K \log\left(\frac{1 + \rho(\beta - 1)}{1 + \frac{\rho}{\bar{\Psi}^\circ}}\right). \quad (44)$$

Although $\bar{\Psi}^\circ$ induces a significant loss in sum-rate, we still have a linear scaling of the sum-rate with SNR [dB] because $\bar{\Psi}^\circ$ is independent of the SNR. Since $\mathbf{\Theta}_\mathrm{T}$ has only a minor impact on R_Δ, for reasonable antenna separations, we obtain for $\mathbf{\Theta}_\mathrm{T} = \mathbf{I}_M$ and $\rho \to \infty$

$$R_\Delta^{\mathrm{lim}} = \lim_{\rho \to \infty} R_\Delta = K \log \mathrm{Tr} \mathbf{L}^{-1}. \quad (45)$$

From (45) we notice that R_Δ^{lim} is solely depending on the distribution of the channel gains $\mathbf{L}$. As an example we suppose that the K users are uniformly distributed on a ring of maximal and minimal radius $r_{\max}$ and $r_{\min}$, respectively. Furthermore, denoting d_k the distance from user k to the transmitter, we apply the exponential path loss model i.e. $l_k = \kappa d_k^{-\alpha}$ where κ is chosen s.t. $E\, l_k = 1$ for given $r_{\max}$, $r_{\min}$ and α. This normalization ensures a fair comparison to the scenario $\mathbf{L} = \mathbf{I}_K$. Assuming $r_{\max} \gg r_{\min}$, we obtain

$$R_\Delta^{\mathrm{lim}} \approx K \log\left(\frac{4}{\alpha^2 - 4}\right) + (\alpha - 2) K \log\left(\frac{r_{\max}}{r_{\min}}\right). \quad (46)$$

Therefore, the sum-rate gap R_Δ^{lim} increases with α and $\log r_{\max}$ for a fixed $r_{\min}$.

VI. Numerical Results

In our simulations all results are averaged over 10,000 independent Rayleigh fading channel realizations. Additional results can be found in [14].

Figure 1 compares our deterministic results to Monte-Carlo simulations for correlated channels with different user path losses. We indicate the standard deviation of the simulations by error bars.

The transmit correlation is assumed to depend only on the distance d_{ij}, $i, j = 1, 2, \ldots, M$ between antennas i and j placed on a uniform circular array (UCA). Thus, $(\mathbf{\Theta}_\mathrm{T})_{ij} = \mathrm{J}_0(2\pi d_{ij}/\lambda)$ [18], where J_0 is the zero-order Bessel function

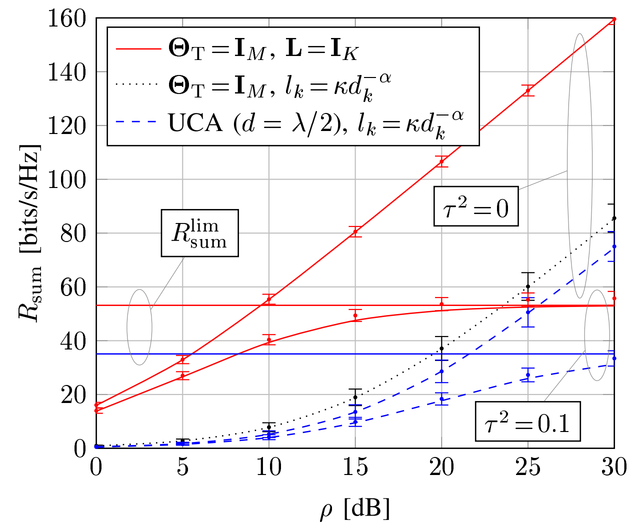

Fig. 1. Ergodic sum-rate vs. SNR with $M = 32$, $\beta = 2$ simulation results are indicated by circle marks with error bars indicating the standard deviation.

of the first kind and λ is the signal wavelength. To assure that $\|\mathbf{\Theta}_\mathrm{T}\|$ is bounded from above we suppose that the distance between adjacent antennas $d = d_{i,i+1}$ is independent of M, i.e. as M grows the radius of the UCA increases.

Furthermore, we consider that the users are distributed uniformly on a ring with $r_{\max} = 500$m and $r_{\max} = 35$m with $\alpha = 3.5$, [19] ("Suburban Macro") and κ s.t. $\mathrm{Tr}\mathbf{L} = 1$.

From Figure 1 we observe, that the expressions derived for large (K, M) lie approximately within one standard deviation of the simulation results even for finite (K, M). To avoid the small divergence of the deterministic equivalents from the simulation results, (K, M) must be increased. For high SNR, the sum-rate loss due to path loss is given by (46), $R_\Delta^{\mathrm{lim}} \approx 75$ [bits/s/Hz], corresponding well to the simulation results.

In Figures 2 and 3, we depict the optimal number of users $K^\star$ that maximize the sum-rate for different system parameters.

Figure 2 compares the optimal number of users $K^\star$ derived from the expressions for asymptotically large (K, M), i.e. Equation (43), to the optimal number of users $K_{\mathrm{mc}}^\star$ obtained from *Monte-Carlo* simulations. More precisely $K_{\mathrm{mc}}^\star$ is the number of users $K < M$ that maximizes the ergodic sum-rate, when distributed uniformly over the ring defined above. Note that we do not perform any user scheduling i.e. we do not test all possible combinations of $K \subseteq M$ since that would alter the effective channel distribution. It can be observed that the optimal number of users $K^\star$ predicted by the asymptotic results do fit well even for finite dimensions. Moreover, introducing correlation and path loss leads to larger dispersion of $K^\star$ over the selected SNR range.

The impact of the number of served users on the ergodic sum-rate is depicted in Figure 3. In the simulations we plot the ergodic sum-rate for an optimal $K_{\mathrm{mc}}^\star$ found by exhaustive search and compare the results with the optimal $K^\star$ obtained from (43). It can be observed that the optimal number of users

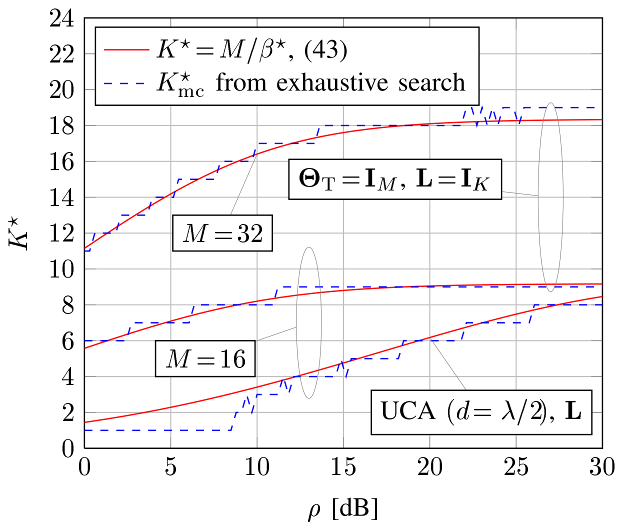

Fig. 2. Sum-rate maximizing ratio $K^\star$ vs. ρ with $\tau^2 = 0.1$.

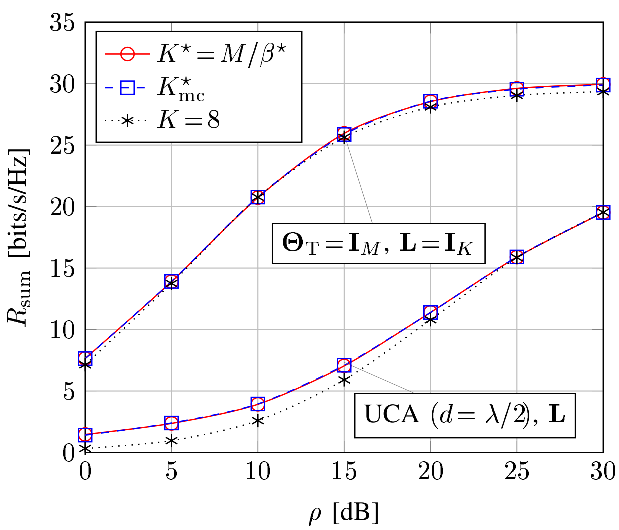

Fig. 3. Ergodic sum-rate vs. average SNR with $M = 16$, $\tau^2 = 0.1$.

$K^\star$ predicted by the asymptotic analysis achieves most of the sum-rate even for finite (K, M) and thus, is a good choice for the user allocation at the transmitter. Moreover, we see that adapting the number of users is beneficial compared to a fixed K. From Figure 2 we identify $K = 8$ as a good choice (for $\tau^2 = 0.1$) and, as expected, the performance is optimal in the medium SNR regime and suboptimal at low and high SNR. The situation changes by adding correlation and path loss. Since $K = 8$ is highly suboptimal for low and medium SNR (cf. Figure 2) we observe a significant loss in sum-rate in this regime.

VII. Conclusion

In this paper we derived an approximation of the sum-rate for ZF precoding in the MISO broadcast channel. This approximation is independent of the channel realizations, asymptotically exact, and is shown to fit well with the numerical results even for small system dimensions. The approximated sum-rate then allowed us to develop expressions for the optimal number of users in the cell and to characterize the impact of a user distribution on the achievable sum-rate.

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