Digital Communications
Exercises

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Exercise 1

Consider a real channel $Y = X + N$, with $N \sim \mathcal{N}(0, N_0)$, $N_0 = 0.1$, and suppose that $\mathrm{E}[X^2] \leq 1$. If the input lies in an L-ary alphabet $A = \{\pm 1c, \pm 3c, \ldots, \pm (L-1)c\}$, how big can be L so that the probability of error is about 10^{-6} using ML detection?

Exercise 1

Consider the probability of error $P(e)$ for PAM modulation

$$P(e) = \frac{2(L-1)}{L} Q\left(\sqrt{\frac{6\mathrm{E}[X^2]}{(L^2-1)N_0}}\right)$$

and determine the largest possible $P(e)$ (using possibly the approximation of the Q function.

MAP detection

Exercise 2

A discrete communication channel with input x and output y has input alphabet $\{0, 1\}$, output alphabet $\{a, b, c, d\}$ and transition probability $P(y|x)$ given by

$$P(a|0) = 0.3, \quad (1)$$
$$P(a|1) = 0.1, \quad (2)$$
$$P(b|0) = 0.1, \quad (3)$$
$$P(b|1) = 0.5, \quad (4)$$
$$P(c|0) = 0.5, \quad (5)$$
$$P(c|1) = 0.1, \quad (6)$$
$$P(d|0) = 0.1, \quad (7)$$
$$P(d|1) = 0.3. \quad (8)$$

The a priori probability of the input is $P(x = 0) = 0.6$ and $P(x = 1) = 0.4$. Compute the MAP detection rule and the resulting average error probability.

Exercise 2

The decision region for 0 is the set of outputs (subset of $\{a, b, c, d\}$), which satisfies

$$P(0|y) > P(1|y).$$

Using Bayes' rule, this is the set of y's that satisfies

$$P(y|0)P(0) > P(y|1)P(1).$$

Call D_0 the detection region for 0 and D_1 the detection region for 1. The probability of error is then

$$P(e) = P(e|0)P(0) + P(e|1)P(1) = P(y \in D_1|0)P(0) + P(y \in D_0|1)P(1).$$

Exercise 3

Consider a real binary input Laplacian noise channel $y = x + n$, with input alphabet $\{-1, +A\}$ and noise pdf

$$p(n) = \frac{b}{2} e^{-b|n|}.$$

The prior input probability distribution is $P(A) = 0.8$ and $P(-A) = 0.2$. Find the MAP decision regions and compute the average decision error probability. Compare to the Gaussian case.

Exercise 3

We proceed similarly as for Exercise 2. Now the error probability is continuous and therefore $P(y \in D_{-A}| + A)$ and $P(y \in D_{+A}| - A)$ are integral forms.
Solution: Two cases must be isolated, depending on whether $A < \log(2)/b$. If so, $\mathcal{D}_A = \mathbb{R}$, otherwise, $\mathcal{D}_A = (-\log(2)/b, \infty)$.

Exercise 4

Consider the signal constellation defined by the points

$$s_m = \sqrt{E_1}\exp\left(i(m-1)/2+\pi/4\right), \text{ for } m = 1,\dots,4 \quad (9)$$

$$s_m = \sqrt{E_2}\exp\left(i(m-5)/2\right), \text{ for } m = 5,\dots,8. \quad (10)$$

1. Find the average energy per symbol E_s (with equiprobabable entries) as a function of E_1 and E_2.
2. For fixed E_s, find the ratio $\rho = E_2/E_1$ that maximizes the minimum squared Euclidean distance of the constellation (assume $E_2 \geq E_1$).
3. For the optimal ratio ρ found, determine an upperbound to the symbol error probability as a function of E_b/N_0.

Exercise 4

1. E_s is defined by

$$E_s = \frac{1}{M}\sum_{m=1}^{M} |s_m|^2.$$

2. The distance $d_{\min}$ between two close points is given by

$$d_{\min} = \|s_1 - s_5\|.$$

   It suffices to write $d_{\min}$ as a function of E_1 alone, find E_1 that reaches the minimum, and determine ρ.

3. Use the fact that error regions for points further than the two points at $d_{\min}$ are included in the error regions for the closer points. Hence

$$P(e) \leq 2Q\left(\sqrt{\frac{\log_2(M)d_{\min}^2}{2}\frac{E_b}{N_0}}\right).$$

Exercise 5

In a 4-QAM modulator, each symbol is labeled by the pair of bits $(b_1 b_2)$ according to the following labeling rule:

$$\mathbf{a}_0 = \sqrt{E}e^{j\pi/4} \mapsto 00, \quad (11)$$

$$\mathbf{a}_1 = \sqrt{E}e^{3j\pi/4} \mapsto 01, \quad (12)$$

$$\mathbf{a}_2 = \sqrt{E}e^{-3j\pi/4} \mapsto 10, \quad (13)$$

$$\mathbf{a}_3 = \sqrt{E}e^{-j\pi/4} \mapsto 11. \quad (14)$$

(15)

The bits b_1 and b_2 are statistically independent with probability $P(b_i = 0) = \varepsilon$, $P(b_1 = 1) = 1 - \varepsilon$, $\varepsilon < 0.5$. The signal is modulated and transmitted over an AWGN channel with power spectral density N_0.
Determine the decision regions of the MAP detector (that minimizes the symbol error probability) and the decision regions of a detector that minimizes the bit error probability.

Exercise 5

The MAP decision regions unfold as in previous exercises. As for the bit error probability, consider individually the first and second bit. The probability of bit error is the average.

Exercise 6

Consider the input equiprobable sequence $a_n \in \{-1, +1\}$, that passes through a convolution channel h_n with $H(z) = 1 + \frac{1}{2}z^{-1}$. The signal is affected by white Gaussian noise, and is received as

$$y_0 = 1.2,\ y_1 = -0.3,\ y_2 = -0.5,\ y_3 = 0.1,\ y_4 = 1.5.$$

Assuming all sequences of $a_0, \dots, a_4$ could be transmitted, what is the sequence detected by a symbol-by-symbol detector? What is the sequence detected by a Viterbi detector?

Exercise 6

Symbol by symbol detection takes merely the sign function as a decision rule: $\hat{a}_n = \mathrm{sign}(y_n)$. Viterbi detection requires to design the state machine where the states are based on the four inputs (a_n, a_{n-1}), but for the first stage where the states are the two inputs of a_0. The channel memory length here is $L = 1$ and the correlator coefficients are $g_0 = 1$, $g_1 = 1/2$.

Exercise 7

$x_1, \ldots, x_N$ are independent samples of a zero mean Gaussian random variable, whose variance is known. Give the expression of the estimated variance provided by ML estimation.

Exercise 7

It suffices to find

$$\arg\max_{\sigma^2} P(x_1, \ldots, x_N | \sigma^2).$$

This is done by remembering that $x_1, \ldots, x_N$ are independent, and therefore $P(x_1, \ldots, x_N | \sigma^2) = \prod_i P(x_i | \sigma^2)$. We then differentiate this expression, equate to zero and determine the maximizing estimate $\hat{\sigma^2}$ of σ^2.

Non-uniform distribution

Exercise 8

Consider a random variable $X \in \{-3\alpha, -\alpha, \alpha, 3\alpha\}$ with a priori probabilities $P(\pm\alpha) = 0.4$, $P(\pm 3\alpha) = 0.1$. The parameter α is set so that the mean signal energy is 1. Given an observation of $Y = X + N$, N being zero mean real Gaussian with variance σ^2, independent on X, what are the MAP decision regions? If $\sigma^2 = 0.25$ and $Y = 2.1$, what is the decision? What is the overall probability of error?

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Exercise 8

Computing $\sum_{i=1}^{4} X_i^2 = 1$, we obtain $\alpha \simeq 0.62$. The decision regions are then given by

$$\mathcal{D}_{-3\alpha} = \left(-\infty, \frac{\sigma^2}{2\alpha}\frac{\log(0.1)}{\log(0.4} - 2\alpha\right]$$

$$\mathcal{D}_{-\alpha} = \left[\frac{\sigma^2}{2\alpha}\frac{\log(0.1)}{\log(0.4} - 2\alpha, 0\right]$$

the other regions being symmetric.