LARGE DIMENSIONAL ASYMPTOTICS OF MULTI-TASK LEARNING

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ABSTRACT

Inspired by human learning, which transfers knowledge from learned tasks to solve new tasks, multitask learning aims at simultaneously solving multiple tasks by a smart exploitation of their similarities. How to relate the tasks so to optimize their performances is however an open problem.

Based on a random matrix approach, this article proposes an asymptotic analysis of a support vector machine-inspired multitask learning scheme. The asymptotic performance of the algorithm, validated on both synthetic and real data, sets forth the relation between the statistics of the data in each task and the hyperparameters relating the tasks together. The article, as such, provides first insights on an offline control of multitask learning, which finds natural connections to the currently popular transfer learning paradigm.

Index Terms— Multitask Learning, Transfer Learning, Random Matrix Theory.

1. INTRODUCTION

The performance of regression and classification tasks can be improved by making the tasks learn from each other. In machine learning, the subject is approached by multitask learning (MTL) where, by appropriately sharing the algorithm parameters, each task may benefit from the others. This has been widely used in face recognition [1], citywide passenger flow prediction [2], cancer survival analysis [3], market preferences [4, 5], to cite a few.

Modelling the relatedness between the tasks is one of the key aspects of MTL. Among the proposed approaches, hierarchical Bayes models [4, 5, 6, 7] are one of the most considered; these assume that the tasks share a common hypothesis class. Natural extensions of standard one-task methods (such as support vector machines) have also been devised [8]. In this precise case, to further cope with the possibly large complexity of training these algorithms, explicit least square alternatives have been proposed [9].

Following up on the asymptotic analysis performed in [10] for least-square support vector machines (in a single task setting), or in [11] for a semi-supervised extension, this article provides a large dimensional analysis of multitask least square support vector machines.

The analysis performed allows to predict the classification error, then providing some insights about the algorithm and an automatic way to tune the hyper parameters.

The remainder of the article is organized as follows. Section 2 introduces the multitask learning setting analyzed in Section 3.2, which provides the asymptotic classification error and discusses the results. Section 4 provides supporting experiments in both synthetic and real data settings, demonstrating a strong consistency with the theoretical findings.

Notation. \( e_m^i \in \mathbb{R}^n \) is the canonical vector with \( e_m^i \mid_i = \delta_{mi} \). Moreover, \( e_{ij} = e_{2k}^{[2k]} \). The notation \( A \otimes B \) for matrices or vectors \( A, B \) is the Kronecker product.

2. SETTING AND ASSUMPTIONS

Let \( X = [X_1, \ldots, X_k] \in \mathbb{R}^{p \times n} \) be the collection of \( n \) independent data vectors drawn from \( k \) "tasks". Task \( i \) is a binary classification problem from the training samples \( X_i = [X_i^{(1)}, X_i^{(2)}] \in \mathbb{R}^{p \times n_i} \) with \( X_i^{(j)} = [x_{i1}, \ldots, x_{in_i}] \in \mathbb{R}^{p \times n_{ij}} \) the \( n_{ij} \) vectors of class \( j \in \{1, 2\} \) for the task \( i \).

The multitask learning least square support vector machine (MTL-LSSVM) aims to predict an output \( y \in \{-1, 1\} \) for any input vector \( x \in \mathbb{R}^p \). To this end, MTL-LSSVM determines \( k \) hyperplanes \( W = [\omega_1, \omega_2, \ldots, \omega_k] \) and a bias term \( b = [b_1, b_2, \ldots, b_k]^T \) that minimize the objective function with constraints

\[
\min_{W \in \mathbb{R}^{kp}, b \in \mathbb{R}^p} J(W, b) = \frac{1}{2} \text{tr}(W^T W) + \frac{1}{2} \sum_{i=1}^k \| \xi_i \|^2,
\]

\[\text{s.t.} \quad y_i = X_i^T \omega_i + b_1 1_{n_i} + \xi_i \quad \forall i \in \{1, \ldots, k\}\]

with \( y_i = [e_i \otimes 1^n_{n_{ij}}, e_i \otimes -1^n_{n_{ij}}]^T \) the label of the training data of task \( i \), and \( \xi_i \in \mathbb{R}^{n_{ij}} \) the errors on the training data. The hyperparameter \( \gamma \) compromises the smallness of the errors \( (\xi_i)_{1 \leq i \leq k} \) against the accuracy of the hyperplanes \( W \).

In order to incorporate the relatedness between tasks, [9] further considers that each hyper plane \( \omega_i \) can be written under the form \( \omega_i = v_i + \omega_0 \), where \( \omega_0 \) carries the “common” information between tasks and \( v_i \) is specialized to each task. The cost function thus becomes

\[
J(\omega_0, V, b) = \frac{1}{2} \| \omega_0 \|^2 + \frac{\lambda}{2k} \text{tr}(V^T V) + \frac{\gamma}{2} \sum_{i=1}^k \| \xi_i \|^2
\]

where the parameter \( \lambda \) enforces the relatedness of the tasks and \( V = [v_1, \ldots, v_k] \). Figure 1 schematically depicts the multitask learning framework.

By introducing the Lagrangian parameter \( \alpha \) and solving the dual formulation of the optimization problem, the solution of (1) is explicit and reads

\[
\omega_i = \left( e_i^{[k]} \otimes I_p \right) A Z^T \alpha
\]

\[\alpha = Q(y - Pb)\]

\[b = (P^T Q P)^{-1} P^T Q y\]
with $y = [y_1, y_2, \ldots, y_T] \in \mathbb{R}^{kn}$, $Q = (\frac{T AZ}{kp} + \frac{1}{kp})^{-1}$, $A = (\frac{2}{n} I_k + \frac{1}{kp}) \otimes I_p$, $Z = \sum_{i=1}^{k} e_i^{(k)} \otimes X_i$, and $P = \sum_{i=1}^{k} e_i^{(k)} \otimes I_{n_i}$. The prediction of the label of any new data point $x \in \mathbb{R}^p$ for task $i$ is then obtained from classification score $g_i(x)$ given by

$$g_i(x) = \frac{1}{kp} \left( e_i^{(k)} \otimes x \right)^T AZ \alpha + b_i. \quad (2)$$

Our objective is to quantify the performance of multitask learning, and thus the (a priori intricate) statistics of $g_i(x)$. To this end, we resort to a large dimensional analysis of $g_i(x)$ in the limit of large $p$ and $n_i$. In order to keep our results valid for very general (random) datasets, we work under the following concentration of measure assumption for $X$.

**Assumption 1 (Distribution of $X$).** There exist two constants $C, c > 0$ (independent of $n, p$) such that, for any $1$-Lipschitz function $f : \mathbb{R}^{p \times n} \rightarrow \mathbb{R}$,

$$\forall t > 0 : \mathbb{P}(|f(X) - \mathbb{E}[f(X)]| \geq t) \leq Ce^{-t^2/c^2}.$$

We further define $E[x_{ij}] = \mu_{ij}$ and $\text{Cov}[x_{ij}] = \Sigma_{ij}$, which only depends on $(i, j)$.

Assumption 1 notably encompasses the following scenarios: $x_{ij} \in \mathbb{R}^p$ are (i) independent Gaussian random vectors with covariance of bounded norm, (ii) independent random vectors uniformly distributed on the $\mathbb{R}^p$ sphere, and most importantly (iii) any Lipschitz transformation $\phi(x_{ij})$ of the above two cases, with bounded Lipschitz norm. Scenario (iii) is particularly relevant to model very realistic data settings as it was recently shown [12] that random data (in particular images) generated by generative adversarial networks (better known as GANs) are by definition concentrated random vectors.

For our analysis, we further place ourselves under the following large $p, n_i$ regime.

**Assumption 2 (Growth Rate).** As $n \rightarrow \infty$, $n_i/p \rightarrow c_0 > 0$ and for $1 \leq i \leq k$, $1 \leq j \leq 2$, $\frac{n_i}{P} \rightarrow c_{ij} > 0$. Besides, $||\mu_{ij}|| = O(\sqrt{P})$.

The assumption on the norm $||\mu_{ij}||$ is quite natural (typically, the norm of the vectors $x_{ij}$ is that as that of their means). Yet, in order to reach non-trivially asymptotic simple results in the following, it will be necessary that $||\mu_{ij} - \mu_{ij'}|| = O(1)$ (i.e., tasks are sufficiently similar).

Under these assumptions, by a careful exploitation of tools from random matrix theory, we can derive the asymptotic distribution of the classification score $g_i(x)$, which is our main result, discussed next.

### 3. MAIN RESULT

#### 3.1. Asymptotic classification error of MTL-LSSVM

Using algebraic identities and the definition of $\alpha$, $g_i(x)$ can be conveniently rewritten as

$$g_i(x) = \frac{1}{kp} \left( e_i^{(k)} \otimes x \right)^T AZQ(y - Pb) + b_i$$

$$= \frac{1}{kp} \left( e_i^{(k)} \otimes x \right)^T A\hat{Q}A\hat{Q}^T Z(y - Pb) + b_i \quad (3)$$

where $\hat{Q} = (A\hat{Q}A\hat{Q}^T + \frac{1}{kp})^{-1}$.

To compute the statistics of $g_i(x)$, we shall resort to determining so-called deterministic equivalents for the matrices $\hat{Q}, A\hat{Q}A\hat{Q}^T Z$, etc., which appear at the core of the formulation of $g_i(x)$. A deterministic equivalent, say $\tilde{F} \in \mathbb{R}^{n \times p}$, of a given random matrix $F \in \mathbb{R}^{n \times p}$, denoted $F \leftrightarrow \tilde{F}$, is defined by the fact that, for any deterministic linear functional $f : \mathbb{R}^{n \times p} \rightarrow \mathbb{R}$, $f(F) \rightarrow \tilde{F} \rightarrow 0$ almost surely (for instance, for $u, v$ of unit norm, $u^T (F - \tilde{F})v \overset{a.s.}{\rightarrow} 0$ and, for $A \in \mathbb{R}^{p \times n}$ deterministic of bounded operator norm, $\frac{1}{n} \text{tr}(F - \tilde{F}) \overset{a.s.}{\rightarrow} 0$). Deterministic equivalents are thus particularly suitable to handle bilinear forms involving the random matrix $F$. Since $g_i(x)$ is precisely a bilinear form involving $A\hat{Q}A\hat{Q}^T Z$, the following lemma is our key technical result.

**Lemma 1 (Deterministic equivalents).** Define, for class $j$ in task $i$ of the data statistics matrices

$$M = (e_i^{(k)} \otimes [\mu_{11}, \mu_{12}], \ldots, e_i^{(k)} \otimes [\mu_{k1}, \mu_{k2}])$$

$$C_{ij} = A\hat{Q}^T \left( e_i^{(k)} e_j^{(k)^T} \otimes (\Sigma_{ij} + \mu_{ij} \mu_{ij}^T) \right) A^\dagger.$$

Then we have the deterministic equivalents of first order

$$\hat{Q} \leftrightarrow \bar{Q} \equiv \left( \sum_{i,j}^{k} \frac{c_{ij}}{c_{0}} \frac{C_{ij}}{1 + \Delta_{ij}} + \frac{1}{7} \right)^{-1}$$

$$Q \leftrightarrow \bar{Q} \equiv \sum_{i,j}^{k} \frac{1}{1 + \Delta_{ij}} \frac{1}{n} \left( -\frac{2}{n} JM_1 \bar{Q}M_1 \right)$$

$$A\hat{Q}A\hat{Q}^T Z \leftrightarrow M^T A\hat{Q}^T A\hat{Q}^T Z \leftrightarrow M\Delta A\Delta M + \mathcal{E} - M\Delta \bar{Q}M\Delta \mathcal{C}$$

and of second order

$$A^\dagger S_i A\hat{Q} \leftrightarrow A_i$$

$$Z^T A\hat{Q}^T A_i A\hat{Q} Z \leftrightarrow M\Delta A\Delta M + \mathcal{E} - M\Delta \bar{Q}M\Delta \mathcal{C}$$

in which we defined

$$\mathcal{C} = \text{diag} \left( \frac{I_{n_1}}{1 + \Delta_{11}}, \ldots, \frac{I_{n_k}}{1 + \Delta_{k2}} \right)$$

$$\mathcal{E} = \sum_{i,j} \text{tr}(C_{ij} A_i A_j e_i e_j^{(k)^T})$$

$$A_{ij} = \bar{Q} A \hat{Q}^T S_i A \hat{Q} + \sum_{i,j} \frac{d_{ij}}{n} \text{tr}(A \hat{Q} C_{ij} \bar{Q}) B_{ij}$$

$$B_{ij} = \bar{Q} C_{ij} \bar{Q} + \sum_{i,j} \frac{2}{n} d_{ij} T_{(i)(j)} \hat{Q} C_{ij} \hat{Q}$$

$$D = \sum_{i,j} d_{ij} e_i^{(k)} e_j^{(k)^T}, \quad d_{ij} = \frac{n_{ij}}{n(1 + \Delta_{ij})^2}$$

**Proof.**
Theorem 1. Under Assumptions 1–2 and the notations of Lemma 1, the probability of classification error of which is, from Theorem 1, using the definition of the score in (3), the mean with $m_{ij}$ and the tasks become independent of each other. In practice, the values of the means and variances can be consistently estimated. Indeed, they mostly involve up-to-2k dimensional vectors of inner products of the means $\mu_{ij}$ or bilinear forms of the covariance matrices $\Sigma_{ij}$; random matrix methods have long been developed to obtain such estimates.

3.2. Discussion about MTL-LSSVM

Theorem 1 is quite involved and seemingly leaves little room to interpretation. We show here that, in simplified settings, interesting intuitions in fact naturally arise.

First, it is interesting to note that the matrix $A = (\frac{1}{k}I_k + \frac{1}{k\ell}I_{\ell}) \otimes I_p$ weighs the constraint of common hyperplane through the term $\frac{1}{k\ell}I_{\ell}$ against the need for isolating tasks through the term $\frac{1}{k}I_k$, with $\lambda$ compromising the two terms.

In particular, letting $\gamma \ll 1$, the matrix $\bar{Q}$ and $\bar{Q}$ (which, as in conventional least-square methods, mostly control the variance of the algorithm) are essentially proportional to identity matrices. Further assuming equal sized data per class and per task, up to a leading constant $\kappa$, the averaged scores $m_{ij}$ simplify as:

$$m_{ij} \approx \kappa \sum_{a=1}^{k} \sum_{b=1}^{2} \left( \frac{2}{\lambda} \delta_{ab} + 1 \right) \mu_{ij}^a \mu_{ab}(-1)^{b-1} + \bar{b}.$$  

As such, $\lambda \gg 1$, $m_{ij}$ is driven by $\sum_{a,b} \mu_{ij}^a \mu_{ab}(-1)^{b-1}$ so that the distance between $\mu_{1a}$ and $\mu_{2a}$ depends on the difference in the projections $\mu_{1a}(\sum_{b} \mu_{2a}^b - \sum_{a} \mu_{2a})$ and $\mu_{2a}(\sum_{b} \mu_{1a}^b - \sum_{a} \mu_{1a})$. This is all the more convenient that the $\mu_{1a}$ (and $\mu_{2a}$) are correctly aligned across $a$: in this case the tasks learn from each other. If instead $\lambda \ll 1$, $\mu_{1a}$ and $\mu_{2a}$ differ by their projections onto $\mu_{1a} - \mu_{2a}$ and the tasks become independent of each other.

Note also interestingly that, if there exists a task $a$ for which $\|\mu_{1a} - \mu_{2a}\| > 1$ and that $\mu_{1a,ij}$ is positive and non-vanishing, then Task $a$ will make the possibly non-trivial Task $i$ much easier (and in fact asymptotically trivial). Exploiting the fact that $J^T y$ leverages the size of each class in each task, the same reasoning holds for any Task $a$ for which $n_{a1}, n_{a2} \gg 1$ and $\mu_{1a,ij}$ is positive and non-vanishing.

4. EXPERIMENTS

4.1. Application to synthetic data

The asymptotic classification error derived in Equation (4) opens the possibility to automatically tune the hyperparameters of the algorithm. We focus here our analysis on $\lambda$ which weighs the relatedness between tasks. As previously mentioned, since the statistics $m_{ij}$ and $\sigma_{ij}^2$ can be asymptotically estimated, the value of $\lambda$ minimizing the probability of error can be estimated (i.e., the algorithm can “auto-tune” the relatedness of tasks) by solving

$$\min_{\lambda > 0} \left( \frac{m_{1a} - m_{2a}}{2\sigma_{1a}} \right), \quad m_{ij} = m_{ij}(\lambda), \quad \sigma_{ij} = \sigma_{ij}(\lambda).$$

We experiment this approach on the following two-task setting $(k = 2)$: $x_{1}\sim N(\pm \mu_1, I_p)$ and $x_{2}\sim N(\pm \mu_2, I_p)$, where $\mu_2 = \beta \mu_1 + \sqrt{1 - \beta^2} \mu^*_1$, where $\mu^*_1$ is any vector orthogonal to $\mu_1$ and $\beta \in [0, 1]$. This setting allows us to tune, through $\beta$, the similarity between tasks.

Figure 2 compares, for different values of $\lambda$, the theoretical and empirical classification errors, and emphasizes the error-minimizing value of $\lambda$. Despite the not-so-large values assumed by $n$ and $p$, a very precise match is reported between the asymptotic theory and the practical experiment, with in particular an accurate estimation of the optimal value for $\lambda$. 
achieved at large \( \mu \) with large numbers of data in the source task significant gains are again a very close match and that, while for very similar tasks kernel (of a least square support vector machine adaptation, and for a linear learning, and indirectly of transfer learning, in the simplified setting the present work provides a first theoretical analysis of multitask (Section 3.2), non-trivial gains are still reached on resembling tasks with \( k \) = 2, using Task 1 as a support, or source, to the targeted Task 2).

It is interestingly observed that theoretical and practical graphs are again a very close match and that, while for very similar tasks with large numbers of data in the source task significant gains are achieved at large \( \lambda \) (thereby corroborating the discussion of Section 3.2), non-trivial gains are still reached on resembling tasks with a non-trivial \( \lambda \), but almost no gain is obtained for too dissimilar tasks.

5. CONCLUDING REMARKS

The present work provides a first theoretical analysis of multitask learning, and indirectly of transfer learning, in the simplified setting of a least square support vector machine adaptation, and for a linear kernel (\( X^T X \)) model. Already in this setting, the asymptotic algorithm performance appears to be non trivial, yet carries several non-trivial insights and opens the possibility of an on-line tuning of the hyperparameters of the model. These insights are key to understanding the (likely more involved) behavior of large dimensional data in refined modern formulations of multitask and transfer learning.

A natural extension of the present analysis notably includes the possibility to involve more elaborate data representations as well as kernel models of data similarity as developed in the asymptotic analysis of [15] and [13]. Similarly, the tools provided in [16] to analyze the performance of implicit (rather than explicit, as in the present work) solutions of optimization problems (standard support vector machines, logistic regression, neural network formulations) will bring the present analysis closer to modern considerations in multitask learning.

6. REFERENCES


