Second-order analysis of the joint SINR distribution in Rayleigh multiple access and broadcast channels

Adrien Pelletier
Telecommunications dpt, Supélec
3 rue Joliot-Curie
91192 Gif sur Yvette, France
adrien.pelletier@ens-cachan.fr

Romain Couillet
Telecommunications dpt, Supélec
3 rue Joliot-Curie
91192 Gif sur Yvette, France
romain.couillet@supelec.fr

Jamal Najim
LIGM, Université Paris-Est
5, Boulevard Descartes
Marne La Vallée, France
najim@univ-mlv.fr

***Abstract*—This article studies the joint distribution of the signal-to-interference-plus-noise ratios (SINR) of the users in Rayleigh multiple access channels and broadcast channels, using large dimensional random matrix theory. Two models are studied: a multiple access channel (MAC) with minimum mean square error (MMSE) decoding, and a broadcast channel with regularized zero-forcing (RZF) precoding. It is shown that, in both models, the empirical distribution of the SINRs of the users behaves asymptotically as a Gaussian, with identified mean and variance. The result is applied to the estimation of the proportion of users in outage for a given target rate. This asymptotic Gaussian behavior can be derived from a theoretical approach based on Stein's method in a random matrix theory context.**

I. Motivation

The use of large dimensional random matrix theory to study multiple input multiple output (MIMO) wireless communication systems, which started with Telatar [1] and then Tse and Hanly [2], made it possible to derive asymptotic expressions for the performance of the wide range of models where the parameters of the system are large. First-order asymptotic expressions have been derived in the case of multiple access channels [2], [3] and broadcast channels [4], which have enabled to study the performance of realistic channel models, and made it possible to compare and optimize different precoding and decoding strategies.

However, when confronted with the problem of comparing the quality of service (QoS) of different users, a first-order approach no longer provides a precise description of the performance distribution of the users with respect to each other. For an acurate description of the QoS, it becomes necessary to study the fluctuations of the rates of users around their limit, in a second-order analysis. Such an approach has been investigated in [5], where a central limit theorem is proved for the SINR of an individual user in a MAC channel with inter-antenna correlation. The work presented here, following a similar motivation, is rather focused on the study of the distribution of the rates of *all users* in the cell. This problem is nontrivial because of the correlations between the users' performance. In the framework of large dimensional random matrix theory, as the system grows large, a decorrelation effect will be shown to occur. The knowledge of the joint user performance can be used to optimize the precoding and/or decoding strategies in order to ensure a maximal QoS for all the users.

In this contribution, we study this performance distribution in a multiple access Rayleigh channel, both in the uplink (MAC channel) with MMSE decoding, and in the downlink (broadcast channel) with RZF precoding.

After describing the two models under study, we will show how our main result, Theorem 1, can be derived from the study of the variances and covariances of the users' SINRs by using a Gaussian approximation result from [6]. Theorem 1 will then be applied to the derivation of a confidence interval for the achievable rates of all the users in the channel.

II. System model

A. Multiple access channel

The first model we study is the case of a Rayleigh MAC channel in the uplink, where a base station with N antennas receives signals from K users equipped with a single antenna each. The signal $\mathbf{u} = (u_1, u_2, \ldots, u_K)^T$ sent by the users towards the base station, supposed to be white Gaussian of covariance $\mathbf{I}_K$, travels through the channel $\mathbf{X} = [\mathbf{x}_1, \mathbf{x}_2, \ldots, \mathbf{x}_K] \in \mathbb{C}^{N\times K}$, which is supposed to be memoryless and without inter-antenna correlation. The entries $\mathbf{X}_{ij}$ of $\mathbf{X}$ are independent and identically distributed (iid) with distribution $\mathcal{CN}(0, 1/N)$, the factor $1/N$ being set to ensure finite channel power irrespective of N. The background noise is modeled as an additive Gaussian noise $\mathbf{w} \in \mathbb{C}^N$ of variance $\sigma^2\mathbf{I}_N$. We consider the case where the signal received at the base station is decoded via an MMSE decoder, assuming full channel state information. In this model, the decoded signal at the base station reads

$$\mathbf{y} = (\mathbf{X}^*\mathbf{X} + \sigma^2\mathbf{I}_K)^{-1}\mathbf{X}^*(\mathbf{X}\mathbf{u} + \mathbf{w}) = \mathbf{X}^*\mathbf{Q}(\mathbf{X}\mathbf{u} + \mathbf{w})$$

with $\mathbf{Q} = (\mathbf{X}\mathbf{X}^* + \sigma^2\mathbf{I}_N)^{-1}$, i.e. the resolvent of $\mathbf{X}\mathbf{X}^*$ evaluated at point $(-\sigma^2)$.

In this model, it can be shown that the SINR s_i of user i ($i \in \{1, \ldots, K\}$) can be written as:

$$s_i = \mathbf{x}_i^*\mathbf{Q}^{[i]}\mathbf{x}_i \tag{1}$$

This work is funded by the DIGICOSME project RESACOL.
The work of Romain Couillet is funded by Newcom#.

where

$$\mathbf{Q}^{[i]} = (\mathbf{X}^{[i]}\mathbf{X}^{[i]*} + \sigma^2\mathbf{I}_N)^{-1}$$
$$\mathbf{X}^{[i]} = [\mathbf{x}_1, \ldots, \mathbf{x}_{i-1}, \mathbf{0}, \mathbf{x}_{i+1}, \ldots, \mathbf{x}_K].$$

B. Broadcast channel

The second model we study is the case of a Rayleigh broadcast channel, which corresponds to the same setup as above except that the K users now send signals towards the N-antenna base station.

In the downlink, the signal $\mathbf{u} = (u_1, u_2, \ldots, u_N)^T$ sent by the base station is precoded as:

$$\mathbf{u} = \sum_{k=1}^{K} r_k \mathbf{g}_k \tag{2}$$

where $\mathbf{r} = (r_1, r_2, \ldots, r_K)^T$ is the signal transmitted by the base station towards the users, supposed to be of covariance $\mathbf{I}_K$, and $\mathbf{G} = [\mathbf{g}_1, \mathbf{g}_2, \ldots, \mathbf{g}_K]$ is the precoding matrix. The channel follows the same modelling assumptions as in the uplink, and can be written $\mathbf{X}^* \in \mathbb{C}^{K\times N}$, with $\mathbf{X} = (\mathbf{x}_1, \mathbf{x}_2, \ldots, \mathbf{x}_K) \in \mathbb{C}^{N\times K}$ being the same matrix of $\mathcal{CN}(0, 1/N)$ entries as defined in the uplink model. The signal received by the users has expression:

$$\mathbf{y} = \mathbf{X}^*\mathbf{u} + \mathbf{w} \tag{3}$$

where the additive Gaussian noise $\mathbf{w} = (w_1, w_2, \ldots, w_K)^T \in \mathbb{C}^K$ has variance $\sigma^2\mathbf{I}_K$. The signal power is normalized to unity, such that the SNR is $1/\sigma^2$ as in the uplink. We assume that the precoding is done by regularized zero-forcing (RZF) [7], with the precoding matrix $\mathbf{G}$ defined as:

$$\mathbf{G} = \xi(\mathbf{H}^*\mathbf{H} + \alpha\mathbf{I}_N)^{-1}\mathbf{H}^* \tag{4}$$
$$= \xi(\mathbf{X}\mathbf{X}^* + \alpha\mathbf{I}_N)^{-1}\mathbf{X} \tag{5}$$

where α is the RZF parameter, and ξ is a normalization constant in order to satisfy the following power constraint, which ensures an average transmit power of N for the signal $\mathbf{u}$ sent by the base station:

$$\mathbb{E}\left[||\mathbf{u}||^2\right] = \mathrm{Tr}(\mathbf{G}^*\mathbf{G}) = N \tag{6}$$

which can be rewritten, setting $\mathbf{Q} = (\mathbf{X}\mathbf{X}^* + \alpha\mathbf{I}_N)^{-1}$, as:

$$\xi^{-2} = \frac{1}{N}\mathrm{Tr}(\mathbf{Q}\mathbf{X}\mathbf{X}^*\mathbf{Q}) \tag{7}$$

The definition for $\mathbf{Q}$ is the same as for the uplink model, except that σ^2 is replaced by α.

The individual components of $\mathbf{y}$ express as:

$$y_i = \xi \sum_{k=1}^{K} [\mathbf{X}^*\mathbf{Q}\mathbf{X}]_{ik}\, r_k + w_i \tag{8}$$

The SINR v_i of the i-th user can thus be written as:

$$v_i = \frac{\xi^2\left(\mathbf{x}_i^*\mathbf{Q}\mathbf{x}_i\right)^2}{\xi^2\sum_{k\neq i}[\mathbf{X}^*\mathbf{Q}\mathbf{X}]_{ik}[\mathbf{X}^*\mathbf{Q}\mathbf{X}]_{ki} + \frac{\sigma^2}{N}} \tag{9}$$

which can be rewritten by using (7) as:

$$v_i = \frac{\left(\mathbf{x}_i^*\mathbf{Q}\mathbf{x}_i\right)^2}{\mathbf{x}_i^*\mathbf{Q}\mathbf{X}^{[i]}\mathbf{X}^{[i]*}\mathbf{Q}\mathbf{x}_i + \frac{\sigma^2}{N}\mathrm{Tr}(\mathbf{Q}\mathbf{X}\mathbf{X}^*\mathbf{Q})} \tag{10}$$

III. Main results

We are interested in the asymptotic behavior of the $\{s_i\}_{i=1}^K$ and the $\{v_i\}_{i=1}^K$ in the large random matrix regime, where $N, K \to \infty$ with $c_N = N/(K-1) \to c \in (0, \infty)$. This regime is the most common in practical cases of such channels, where the number of antennas and the number of users are both large and of the same order of magnitude.

Our main result is Theorem 1 below, which shows that the expression for the joint SINR distribution in both models presented above is asymptotically Gaussian. We first introduce Theorem 1 without proof in the following section, and the methodology of the proof will then be exposed in the section after.

A. Notations and main result

Define:

$$m_N^{(1)}(z) = \frac{c_N - 1}{2c_N z} - \frac{1}{2} + \frac{\sqrt{(1 - c_N + c_N z)^2 + 4c_N^2 z}}{2c_N z} \tag{11}$$

and for all $k \geq 1$

$$m_N^{(k)}(z) = \frac{-1}{(k-1)!}\frac{d^{k-1}m_N^{(1)}(z)}{dz^{k-1}} \tag{12}$$

All the $m_N^{(k)}(z)$ converge to a finite nonzero limit as $N, K \to \infty$.

Let Ω be the probability space of sequences ω of random matrices $\mathbf{X}_{N_i,K_i} \in \mathbb{C}^{N_i\times K_i}$ of increasing size: $\omega = \{\mathbf{X}_{N_1,K_1}, \mathbf{X}_{N_2,K_2}, \ldots\}$, with $N_i, K_i \to \infty$ and $N_i/(K_i - 1) \to c$ as $i \to \infty$.

We then define the empirical distributions of the SINRs in the two models:

Multiple access channel: Define the centered and rescaled versions of the uplink SINRs $\overset{\circ}{s}_i$ (taken for $z = \sigma^2$) as:

$$\overset{\circ}{s}_i = \sqrt{\frac{N}{m_N^{(2)}}}(s_i - m_N^{(1)}) \tag{13}$$

and define their empirical distribution $\lambda_{K,\omega}$ as the random probability measure

$$\lambda_{K,\omega} = \frac{1}{K}\sum_{k=1}^{K}\delta_{\overset{\circ}{s}_k(\omega)} \tag{14}$$

Broadcast channel: Define the centered and rescaled versions of the downlink SINRs $\overset{\circ}{v}_i$ (taken for $z = \alpha$) as:

$$\overset{\circ}{v}_i = \sqrt{\frac{N}{\tau_N}}(v_i - \mu_N) \tag{15}$$

with

$$\mu_N = \frac{{m_N^{(1)}}^2}{m_N^{(1)} - \alpha m_N^{(2)} + \left(1 + m_N^{(1)}\right)^2\left(\sigma^2 m_N^{(1)} - \alpha\sigma^2 m_N^{(2)}\right)}$$

$$\tau_N = \left(d_{1,N}^2 m_N^{(2)} + d_{2,N}^2 m_N^{(4)} + 2d_{1,N}d_{2,N}m_N^{(3)}\right)$$

$$d_{1,N} = \frac{m_N^{(1)}\left[m_N^{(1)} - 2\alpha m_N^{(2)} + 2(1+m_N^{(1)})(\sigma^2 m_N^{(1)} - \alpha\sigma^2 m_N^{(2)})\right]}{\left[m_N^{(1)} - \alpha m_N^{(2)} + \left(1+m_N^{(1)}\right)^2\left(\sigma^2 m_N^{(1)} - \alpha\sigma^2 m_N^{(2)}\right)\right]^2}$$

$$d_{2,N} = \frac{\alpha {m_N^{(1)}}^2}{\left[m_N^{(1)} - \alpha m_N^{(2)} + \left(1+m_N^{(1)}\right)^2\left(\sigma^2 m_N^{(1)} - \alpha\sigma^2 m_N^{(2)}\right)\right]^2}$$

and define their empirical distribution ν_K as the random probability measure

$$\nu_{K,w} = \frac{1}{K}\sum_{k=1}^{K}\delta_{\mathring{v}_k(\omega)}. \tag{16}$$

Then the following theorem holds:

Theorem 1. *In the regime K, $N \to \infty$, with $c_N = N/(K-1) \to c \in (0,\infty)$, $\lambda_{K,\omega}$ and $\nu_{K,\omega}$ converge to $\mathcal{N}(0,1)$ for almost every $\omega \in \Omega$.*

Theorem 1 asserts that the empirical distribution of the SINRs of a MAC channel behaves asymptotically like a Gaussian of mean $m_N^{(1)}$ and variance $m_N^{(2)}/N$, and that the empirical distribution of the SINRs of a broadcast channel behaves asymptotically like a Gaussian of mean μ_N and of variance τ_N/N. This asymptotic result is compared to simulations in Figure 1, which shows a good correspondance between asymptotic results and simulation results for parameters typical for a massive MIMO channel.

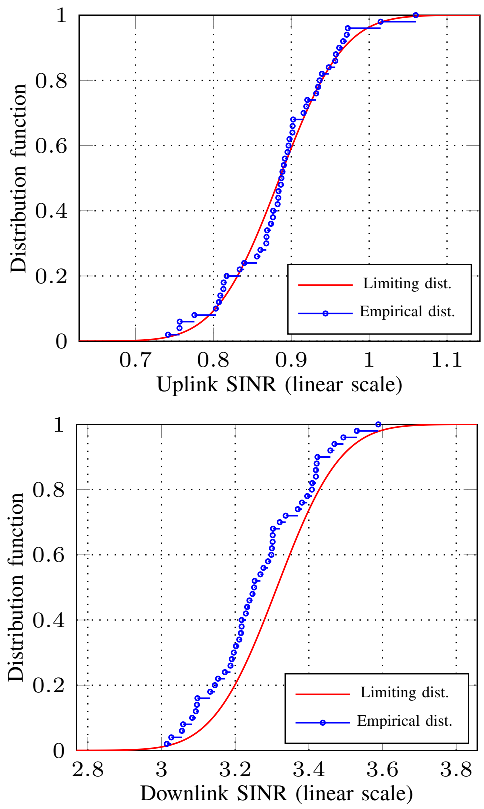

Figure 1. Simulation results for the empirical distributions of the downlink SINRs $\{s_i\}_{i=1}^K$ (top) and uplink SINRs $\{v_i\}_{i=1}^K$ (bottom), compared to the limiting expressions given by Theorem 1. The empirical distribution is evaluated for a *single* generation of the random matrix $\mathbf{X}$. Both graphs are drawn for $N = 200$, $K = 50$, $\sigma^2 = 1$, and $\alpha = \sigma^2/c$.

B. Methodology

The proof of Theorem 1 relies on the Gaussian approximation techniques of Stein's method, revisited by Chatterjee [6] in a random matrix theory context. This section is devoted to the methodology used to prove this result.

The study of the flustuations of the SINRs in the large random matrix regime is done in three steps. The first step consists in computing asymptotic expressions for the means, variances and covariances of the SINRs. The second is to prove a central limit theorem (CLT) giving the convergence of the vector of a finite number of SINRs towards a Gaussian vector of the same mean and variance, by using an implicit method based on [6]. Finally, the third step consists in using the previous CLT to prove the convergence of the empirical distribution of the SINRs towards a Gaussian distribution with the same parameters.

1) Covariance structure: We start by giving the asymptotic covariance structure of the $\{s_i\}$ and $\{v_i\}$.

Since the entries of $\mathbf{X}$ are Gaussian, these terms can be evaluated with the help of Gaussian calculation tools, namely the integration by parts and the Poincaré-Nash inequality [8]:

Multiple acces channel: For the $\{s_i\}$, we have the following result:

Proposition 1. *The first two moments of the vector of the $\{s_i\}$ have the following expression for all $i, j \in \{1,\dots,K\}$ such that $j \neq i$,*

$$\mathbb{E}[s_i] = m_N^{(1)} + \mathcal{O}\left(\frac{1}{N^2}\right) \quad \mathrm{Var}(s_i) = \frac{1}{N}m_N^{(2)} + \mathcal{O}\left(\frac{1}{N^2}\right)$$
$$\mathrm{Cov}(s_i, s_j) = \mathcal{O}\left(\frac{1}{N^2}\right)$$

In the above, $m_N^{(1)}$ and $m_N^{(2)}$ are defined by (11) and (12). The terms $\mathcal{O}\left(\frac{1}{N^2}\right)$ in fact depend only on the asymptotic behavior of N, and do not depend on K.

We will show in the next step how this expression allows to prove a CLT on the $\{s_i\}$.

Broadcast channel: For the $\{v_i\}$, instead of calculating directly the covariance structure of these terms, we can show that v_i has the following decomposition:

$$v_i = \frac{\alpha_i^2}{\alpha_i - \alpha\beta_i + (1+\alpha_i)^2\,\delta} \tag{17}$$

where we defined α_i, β_i and δ as:

$$\alpha_i = \mathbf{x}_i^*\mathbf{Q}^{[i]}\mathbf{x}_i \tag{18}$$
$$\beta_i = \mathbf{x}_i^*{\mathbf{Q}^{[i]}}^2\mathbf{x}_i \tag{19}$$
$$\delta = \frac{\sigma^2}{N}\mathrm{Tr}(\mathbf{QXX}^*\mathbf{Q}). \tag{20}$$

The covariance structure of the terms $\{\alpha_i\}$, $\{\beta_i\}$ and δ is significantly simpler to evaluate than the covariance structure of the $\{v_i\}$. Using Gaussian calculation tools, we have:

Proposition 2. *For all $i, j \in \{1, \ldots, K\}$ such that $j \neq i$,*

$$\begin{array}{ll}
\mathbb{E}[\alpha_i] = m_N^{(1)} + \mathcal{O}\left(\frac{1}{N^2}\right) & \mathbb{E}[\beta_i] = m_N^{(2)} + \mathcal{O}\left(\frac{1}{N^2}\right) \\
\mathrm{Var}(\alpha_i) = \frac{m_N^{(2)}}{N} + \mathcal{O}\left(\frac{1}{N^2}\right) & \mathrm{Var}(\beta_i) = \frac{m_N^{(4)}}{N} + \mathcal{O}\left(\frac{1}{N^2}\right) \\
\mathbb{E}[\delta] = \sigma^2 m_N^{(1)} - \alpha\sigma^2 m_N^{(2)} + \mathcal{O}\left(\frac{1}{N}\right) & \mathrm{Var}(\delta) = \mathcal{O}\left(\frac{1}{N^2}\right) \\
\mathrm{Cov}(\alpha_i, \beta_i) = \frac{m_N^{(3)}}{N} + \mathcal{O}\left(\frac{1}{N^2}\right) & \mathrm{Cov}(\alpha_i, \delta) = \mathcal{O}\left(\frac{1}{N^{3/2}}\right) \\
\mathrm{Cov}(\beta_i, \delta) = \mathcal{O}\left(\frac{1}{N^{3/2}}\right) & \mathrm{Cov}(\alpha_i, \alpha_j) = \mathcal{O}\left(\frac{1}{N^2}\right) \\
\mathrm{Cov}(\beta_i, \beta_j) = \mathcal{O}\left(\frac{1}{N^2}\right) & \mathrm{Cov}(\alpha_i, \beta_j) = \mathcal{O}\left(\frac{1}{N^2}\right)
\end{array}$$

We will show in the next step how this expression allows to prove a CLT on $\{\alpha_i\}$, $\{\beta_i\}$ and δ. This CLT will then be transferred to a CLT the v_i by using the delta-method [9].

2) Central limit theorem for a finite number of users: The above covariance structure results allow to prove a CLT on both the $\{s_i\}$ and the $\{v_i\}$, showing that these two quantities show Gaussian fluctiations around their limit. Since K goes to infinity, however, the CLT will give the asymptotic behavior of a vector of $\{s_i\}$ or $\{v_i\}$ of *finite* size $\hat{K}$. This step details these two results and their derivation.

Multiple acces channel: The asymptotic behavior of a finite number $\hat{K}$ of $\{s_i\}$ is given by:

Theorem 2. *Let $\widehat{K}$ be a fixed finite integer. As $N \to \infty$, the random vector $(s_i)_{i=1,\ldots,\widehat{K}}$ satisfies the following central limit theorem:*

$$\sqrt{\frac{N}{m_N^{(2)}}} \begin{pmatrix} s_1 - m_N^{(1)} \\ s_2 - m_N^{(1)} \\ \cdots \\ s_{\hat{K}} - m_N^{(1)} \end{pmatrix} \xrightarrow{d} \mathcal{N}_{\hat{K}}(\mathbf{0}, \mathbf{I}_{\hat{K}}).$$

Sketch of proof: By the Cramer-Wold device, the study of the random vector $(s_i)_{i=1,\ldots,\widehat{K}}$ of size $\hat{K}$ can be reduced to the study of a scalar random variable equal to an arbitrary linear combination of the s_i. Thus, to prove the theorem, it suffices to prove that $f(\mathbf{X}) = \sum_{i=1}^{\hat{K}} t_i s_i$ for arbitrary t_i converges to a univariate Gaussian.

This univariate convergence is proven by using a Gaussian approxmiation result by Chatterjee [6], which constitutes an adaptation of Stein's method in a random matrix theory context. This result allows to show the convergence in total variation of $f(\mathbf{X})$ to a univariate Gaussian. This method relies on bounds of the first and second derivatives of $f(\mathbf{X})$ with respect to the entries of the random matrix $\mathbf{X}$, and shows the convergence of $f(\mathbf{X})$ to a Gaussian of the same mean and variance. This approach is implicit and does not assume knowledge of the mean and variance of $f(\mathbf{X})$, which are calculated independently as a consequence of the covariance structure of $(s_i)_{i=1,\ldots,\widehat{K}}$.

Finally, the convergence of the moments shown in the previous step allows to replace in the asymptotic regime the mean and variance of this Gaussian approximation by the mean and variance from Prop. 1, which proves Theorem 2. □

Broadcast channel: The asymptotic behavior of a finite number $\hat{K}$ of $\{v_i\}$ is given by:

Theorem 3. *Let $\widehat{K}$ be a fixed finite integer. As $N \to \infty$, the random vector $(v_i)_{i=1,\ldots,\widehat{K}}$ satisfies the following central limit theorem:*

$$\sqrt{\frac{N}{\tau_N}} \begin{pmatrix} v_1 - \mu_N \\ v_2 - \mu_N \\ \vdots \\ v_{\widehat{K}} - \mu_N \end{pmatrix} \xrightarrow{d} \mathcal{N}_{\widehat{K}}(\mathbf{0}, \mathbf{I}_{\widehat{K}}) \tag{21}$$

Sketch of proof: Similarily to the proof of the CLT on the $\{s_i\}$, we use the Cramer-Wold device and Gaussian approximation by Stein's method on the random vector $\{\alpha_i, \beta_j, \delta\}_{i,j=1\ldots\hat{K}}$. This way, we show that this vector of size $2\hat{K}+1$ converges to a Gaussian vector whose means, variances and covariances are those given in the previous step.

By the delta-method, this CLT on $\{\alpha_i, \beta_j, \delta\}_{i,j=1\ldots\hat{K}}$ can be transferred to a CLT on the random vector $(v_i)_{i=1,\ldots,\widehat{K}}$ by using the expression (17) which gives the expressions of μ_N and τ_N and therefore proving Theorem 3. □

We also mention that Theorems 2 and 3 are also valid in the asymptotic regime where $N \to \infty$ and K remains finite, in which case taking $\hat{K} = K$ gives the asymptotic behavior of the whole vector of the SINRs. Therefore these results are also valid in the regime where only the number of antennas is large.

3) Derivation of the asymptotic SINR distribution: The proof of Theorem 1 is based on Theorems 2 and 3. Since the proof is the same for the theorem on the $\{s_i\}$ and the theorem on the $\{v_i\}$, we give only the sketch of the proof for the $\{s_i\}$.

Let $\phi_{K,\omega}(u) = \frac{1}{K}\sum_{k=1}^{K} e^{iu\overset{\circ}{s}_k(\omega)}$ be the characteristic function of $\lambda_{K,w}$. Using the TCL (Theorem 2) for $\hat{K} = 1$ and $\hat{K} = 2$ yields the following approximations of the first two moments of $\phi_{K,\omega}(u)$ for all u:

$$\mathbb{E}\left[\phi_{K,\omega}(u)\right] = e^{-u^2/2} + \mathcal{O}\left(\frac{1}{K^{1/4}}\right) \tag{22}$$

$$\mathrm{Var}(\phi_{K,\omega}(u)) = \mathcal{O}\left(\frac{1}{K^{1/4}}\right) \tag{23}$$

By Chebychev's inequality, (22) and (23) give the convergence in distribution of $\phi_{K,\omega}(u)$ to the standard Gaussian characteristic function for all u. This implies the convergence in probability of $\lambda_{K,w}$ to $\mathcal{N}(0,1)$. We then remark that the subsequence $\phi_{\alpha_n,\omega}(u)$ with $\alpha_n = \lfloor \alpha^n \rfloor$ converges almost surely to $\mathcal{N}(0,1)$ for all $\alpha > 1$. We conclude by bounding the difference between $\phi_{K,\omega}(u)$ and $\phi_{\alpha_n,\omega}(u)$, and taking the limit $\alpha \to 1$. This proves Theorem 1.

The application of Theorem 1 to study the asymptotic behavior of the achievable rates is the object of the next section.

IV. Application to the distribution of user rates

Theorem 1 can be applied to the computation of the distribution of the individual user rates. In both models with Gaussian-distributed signals, the maximum achievable rate of user i is $R_i = \log(1 + s_i)$ in the MAC channel and $\tilde{R}_i = \log(1 + v_i)$ in the broadcast channel. We have the two following results:

Multiple access channel: Let $S_{K,\omega}(R)$ be the distribution function of the rates

$$S_{K,\omega}(R) = \frac{1}{K}\sum_{k=1}^{K} \mathbb{1}_{\{\log(1+s_k(\omega))\leq R\}}. \tag{24}$$

The following result is an application of Theorem 1 to the computation of the quantiles of the user rates.

Corollary 1. *Let $R_K(\sigma^2, q)$ be defined as*

$$R_K(\sigma^2, q) = \log\left(1 + m_N^{(1)} + \sqrt{\frac{m_N^{(2)}}{N}}\Phi^{-1}(q)\right) \tag{25}$$

with Φ^{-1} the inverse of the Gaussian distribution function $\Phi(x) = 1/\sqrt{2\pi}\int_{-\infty}^{x}\exp(-u^2/2)du$. Then, as $N, K \to \infty$ with $c_N \to c \in (0, \infty)$, for almost every ω,

$$S_{K,\omega}\left(R_K(\sigma^2, q)\right) \xrightarrow{a.s} q. \tag{26}$$

Corollary 1 states in particular that, at target rate $R_K(\sigma^2, q)$, for N, K large enough, an approximate proportion q of the users is in outage. The quantile function $R_K(\sigma^2, q)$ is represented in Figure 2 for different values of q. From this figure, we see that, even for a large value of N (N=200 here), there is a significant spread of the SINRs around their first-order asymptotic value (attained for $q = 0.5$).

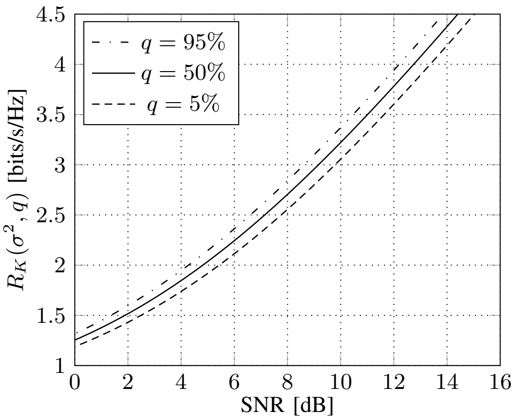

Figure 2. Asymptotic values for quantiles $q = 0.05, 0.5, 0.95$ of the users' rates in a MAC channel given by Corollary 1, versus the signal-to-noise ratio (SNR) $1/\sigma^2$. Here, $N = 200$ and $K = 50$.

Broadcast channel: Let $V_{K,\omega}(R)$ be the distribution function of the rates

$$V_{K,\omega}(R) = \frac{1}{K}\sum_{k=1}^{K} \mathbb{1}_{\{\log(1+v_k(\omega))\leq R\}}. \tag{27}$$

Then, similarly,

Corollary 2. *Let $\tilde{R}_K(\sigma^2, \alpha, q)$ be defined as*

$$\tilde{R}_K(\sigma^2, \alpha, q) = \log\left(1 + \mu_N + \sqrt{\frac{\tau_N}{N}}\Phi^{-1}(q)\right) \tag{28}$$

with Φ^{-1} the inverse of the Gaussian distribution function. Then, as $N, K \to \infty$ with $c_N \to c \in (0, \infty)$, for almost every ω,

$$V_{K,\omega}\left(\tilde{R}_K(\sigma^2, \alpha, q)\right) \xrightarrow{a.s} q. \tag{29}$$

Corollary 2 states in particular that, at target rate $\tilde{R}_K(\sigma^2, \alpha, q)$, for N, K large enough, an approximate proportion q of the users is in outage.

V. Conclusion

We have shown an approach that allowed to derive the performance distribution of the users in MAC and broadcast channels. Such results can be interesting in the optimization of precoding and decoding strategies in the fact that it would enable to optimize not only the mean performance, but also the performance of the users with the lowest QoS in the cell, therefore effectively guaranteeing a minimal QoS for all the users.

We believe that the generality of the theoretical approach used here could allow a generalization of these results to more complex channel models. In particular, it should be possible to take into account effects such as inter-antenna correlation, Rician fading channels, or different pathloss distances between users. Such tools could also prove to be of interest in a multi-cell approach, taking into account the additional interference from neighboring cells.

References

[1] E. Telatar, "Capacity of multi-antenna gaussian channels," *European Transactions on Telecommunications*, vol. 10, no. 6, pp. 585–595, 1999.

[2] D. Tse and S. Hanly, "Linear multiuser receivers : Effective interference, effective bandwidth and user capacity," *IEEE Transactions on Information Theory*, vol. 45, no. 2, 1999.

[3] R. Couillet, M. Debbah, and J. W. Silverstein, "A deterministic equivalent for the analysis of correlated mimo multiple access channels," *Information Theory, IEEE Transactions on*, vol. 57, no. 6, pp. 3493–3514, 2011.

[4] S. Wagner, R. Couillet, M. Debbah, and D. T. Slock, "Large system analysis of linear precoding in correlated miso broadcast channels under limited feedback," *Information Theory, IEEE Transactions on*, vol. 58, no. 7, pp. 4509–4537, 2012.

[5] A. Kammoun, M. Kharouf, W. Hachem, and J. Najim, "A central limit theorem for the SINR at the LMMSE estimator output for large dimensional signals," *IEEE Transactions on Information Theory*, vol. 55, no. 11, 2009.

[6] S. Chatterjee, "Fluctuations of eigenvalues and second-order Poincaré inequalities," *Probability Theory and Related Fields*, vol. 143, pp. 1–40, 2009.

[7] C. B. Peel, B. M. Hochwald, and A. L. Swindlehurst, "A vector-perturbation technique for near-capacity multiantenna multiuser communication-part i: channel inversion and regularization," *Communications, IEEE Transactions on*, vol. 53, no. 1, pp. 195–202, 2005.

[8] L. A. Pastur and M. Shcherbina, *Eigenvalue distribution of large random matrices*. Amer Mathematical Society, 2011, vol. 171.

[9] A. W. Van der Vaart, *Asymptotic statistics*. Cambridge university press, 2000, vol. 3.