Projet RMT4GRAPH

Evaluation à mi-parcours

Romain COUILLET
(en collaboration avec Hafiz Tiomoko Ali)

CentraleSupélec (Paris, France)

9 septembre 2016

CentraleSupélec

Outline

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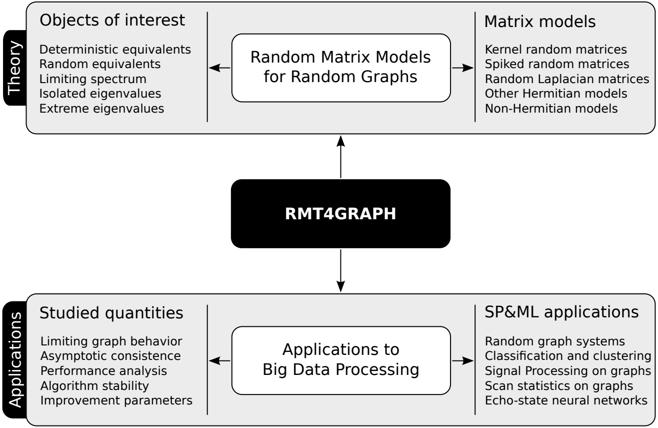
Project definition and objectives

"develop a framework of big data processing analysis (notably graph-based methods) relying on random matrix tools"



Work Packages and Timeline

- **WP1. Random Matrix Models for Random Graphs.**
  - Task 1.1. Kernel random matrix models.
  - Task 1.2. Hermitian models and spikes.
  - Task 1.3. Random matrices with non-linear or recursive entries
    (formerly "Task 1.3. Non-hermitian random matrix models)

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- **WP2. Applications to Big Data Processing.**
  - Task 2.1. Applications to machine learning.
  - Task 2.2. Signal processing on graphs
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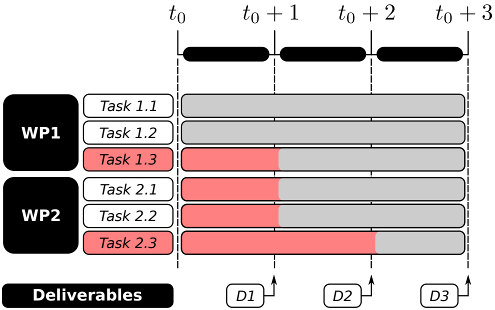

Taskforce and Actions

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Actions and Publications.

- **Publications**: 3 journal articles, 7 conference articles
- **Dissemination**:
  - Organization of the summer school "Large Random Matrices and High Dimensional Statistical Signal Processing", Telecom ParisTech, June 7-8, 2016.
  - SSP'16 Special Session "Random matrices in signal processing and machine learning"
  - Distinguished keynote speaker at EUSIPCO 2016
  - Special Issue on Random Matrices in "Revue du Traitement du Signal"
  - Several invited talks and contributions to local events

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Assume n-node, m-edge graph G, with

- "intrinsic" average connectivity $q_1, \ldots, q_n \sim \mu$ i.i.d.

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- k classes $\mathcal{C}_1, \ldots, \mathcal{C}_k$ independent of $\{q_i\}$ of (large) sizes $n_1, \ldots, n_k$, with preferential attachment C_{ab} between $\mathcal{C}_a$ and $\mathcal{C}_b$

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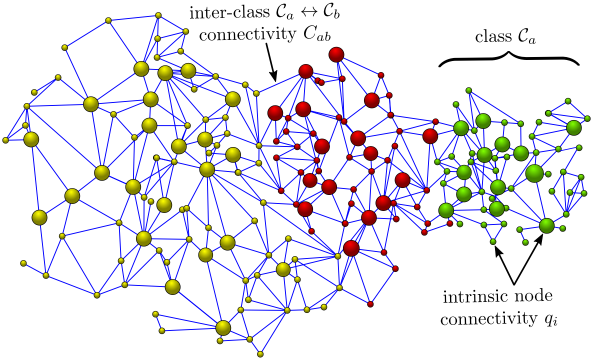
- induces edge probability for node $i \in \mathcal{C}_a$, $j \in \mathcal{C}_b$,

$$P(i \sim j) = q_i q_j C_{ab}.$$



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Understand and improve performance of spectral community detection methods:

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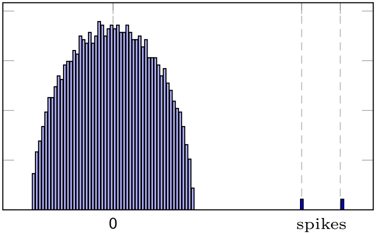

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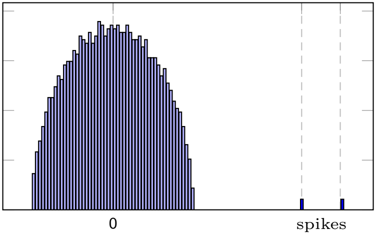
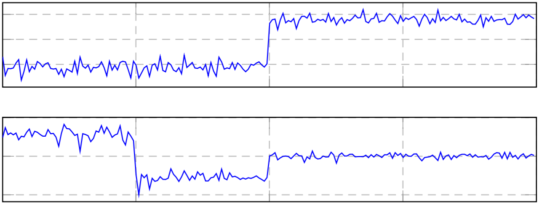


⇓ **Eigenvectors** ⇓



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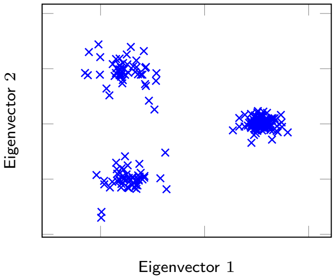

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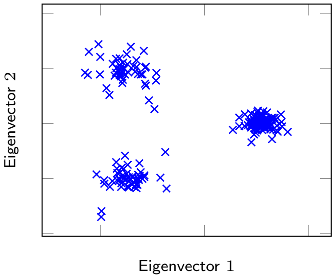

⇓

EM or k-means clustering.

Limitations of Adjacency/Modularity Approach

Scenario: 3 classes with μ bi-modal (e.g., $\mu = \frac{3}{4}\delta_{0.1} + \frac{1}{4}\delta_{0.5}$)

$\rightarrow$ Leading eigenvectors of A (or modularity $A - \frac{dd^{\mathsf{T}}}{2m}$) biased by q_i distribution.
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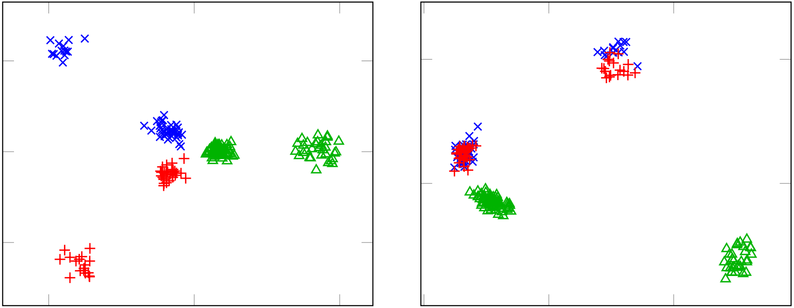

(Modularity) (Bethe Hessian)

Connectivity Model: $P(i \sim j) = q_i q_j C_{ab}$ for $i \in \mathcal{C}_a$, $j \in \mathcal{C}_b$.

Dense Regime Assumptions: Non trivial regime when, as $n \to \infty$,

$$C_{ab} = 1 + \frac{M_{ab}}{\sqrt{n}}$$

with $M_{ab} = O(1)$ (fixed).

Regularized Modularity Approach

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Considered Matrix:
For $\alpha \in [0, 1]$, (and with $D = \mathrm{diag}(A1_n) = \mathrm{diag}(d)$ the degree matrix)

$$L_\alpha = (2m)^\alpha \frac{1}{\sqrt{n}} D^{-\alpha} \left[A - \frac{dd^\mathsf{T}}{2m} \right] D^{-\alpha}.$$

Asymptotic Equivalence

Theorem (Limiting Random Matrix Equivalent)

For each $\alpha \in [0,1]$, *as* $n \to \infty$, $\|L_\alpha - \tilde{L}_\alpha\| \to 0$ *almost surely, where*

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$$\tilde{L}_\alpha = \frac{1}{\sqrt{n}} D_q^{-\alpha} X D_q^{-\alpha} + U \Lambda U^\mathsf{T}$$

with $D_q = \operatorname{diag}(\{q_i\})$, X *zero-mean random matrix,*

$$U = \left[D_q^{1-\alpha} \frac{J}{\sqrt{n}} \quad \frac{1}{1_n^\mathsf{T} D_q 1_n} D_q^{-\alpha} X 1_n \right], \quad \textit{rank } k+1$$

$$\Lambda = \begin{bmatrix} (I_k - 1_k c^\mathsf{T}) M (I_k - c 1_k^\mathsf{T}) & -1_k \\ 1_k^\mathsf{T} & 0 \end{bmatrix}$$

and $J = [j_1, \ldots, j_k]$, $j_a = [0, \ldots, 0, 1_{n_a}^\mathsf{T}, 0, \ldots, 0]^\mathsf{T} \in \mathbb{R}^n$ *canonical vector of class* $\mathcal{C}_a$.

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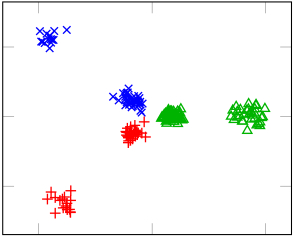
- it helps us correct and optimize classical spectral clustering into a powerful **new algorithm**.

Performance Results (2 masses of q_i)



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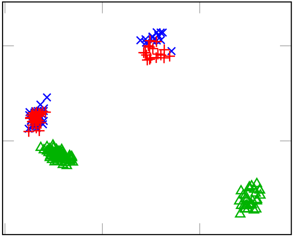

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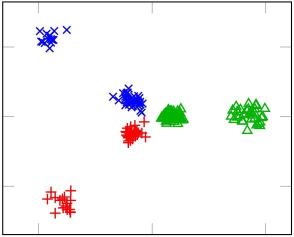

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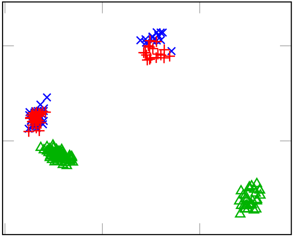

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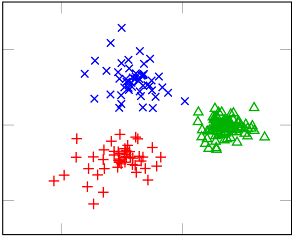

(Algo with $\alpha = 1$)

Figure: Two dominant eigenvectors (x-y axes) for $n = 2000$, $K = 3$, $\mu = \frac{3}{4}\delta_{q_1} + \frac{1}{4}\delta_{q_2}$, $q_1 = 0.1$, $q_2 = 0.5$, $c_1 = c_2 = \frac{1}{4}$, $c_3 = \frac{1}{2}$, $M = 100 I_3$.

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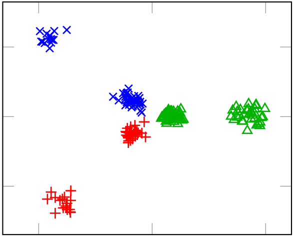

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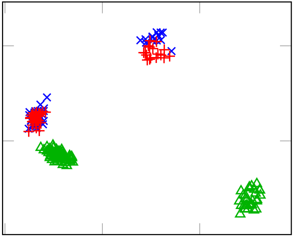

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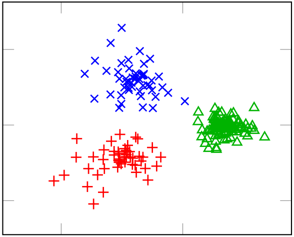

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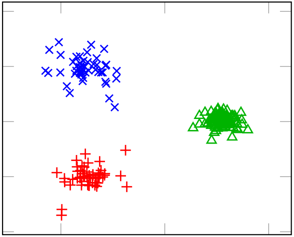

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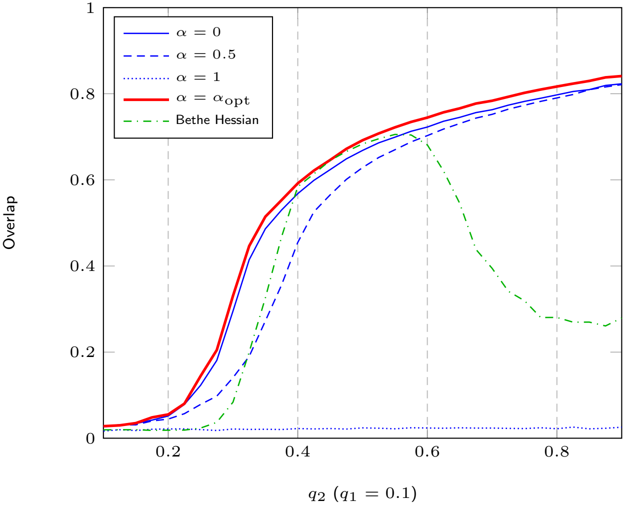

Figure: Overlap performance for $n = 3000$, $K = 3$, $\mu = \frac{3}{4}\delta_{q_1} + \frac{1}{4}\delta_{q_2}$ with $q_1 = 0.1$ and $q_2 \in [0.1, 0.9]$, $M = 10(2I_3 - 1_3 1_3^{\mathsf{T}})$, $c_i = \frac{1}{3}$.

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for some similarity kernel $\kappa(x, y) \geq 0$ (large if x similar to y).

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where $\mathcal{M} \subset \mathbb{R}^{n \times k} \cap \left\{M; \ M_{ij} \in \{0, |\mathcal{S}_j|^{-\frac{1}{2}}\}\right\}$ (in particular, $M^{\mathsf{T}}M = I_k$) and

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- But integer problem! Usually NP-complete.

Towards kernel spectral clustering

- Kernel spectral clustering: discrete-to-continuous relaxations of such metrics

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i.e., eigenvector problem:

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- Refinements:
  - working on K, $D-K$, $I_n - D^{-1}K$, $I_n - D^{-\frac{1}{2}}KD^{-\frac{1}{2}}$, etc.
  - several steps algorithms: Ng–Jordan–Weiss, Shi–Malik, etc.

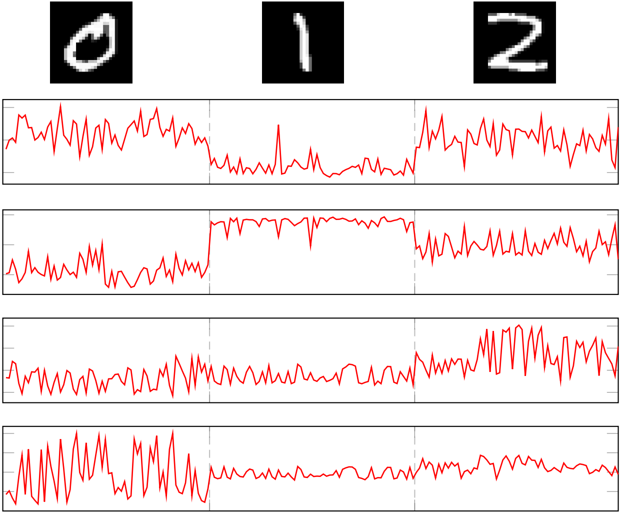

Figure: Leading four eigenvectors of $D^{-\frac{1}{2}} K D^{-\frac{1}{2}}$ for MNIST data.

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Methodology and objectives

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Methodology:

- Use statistical assumptions (Gaussian mixture)
- Benefit from doubly-infinite independence and random matrix tools

Model and Assumptions

Gaussian mixture model:

- $x_1, \ldots, x_n \in \mathbb{R}^p$,
- k classes $\mathcal{C}_1, \ldots, \mathcal{C}_k$,
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- We study the normalized Laplacian:

$$L = nD^{-\frac{1}{2}} K D^{-\frac{1}{2}}$$

  with $D = \mathrm{diag}(K 1_n)$.

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Theoretical Findings versus MNIST

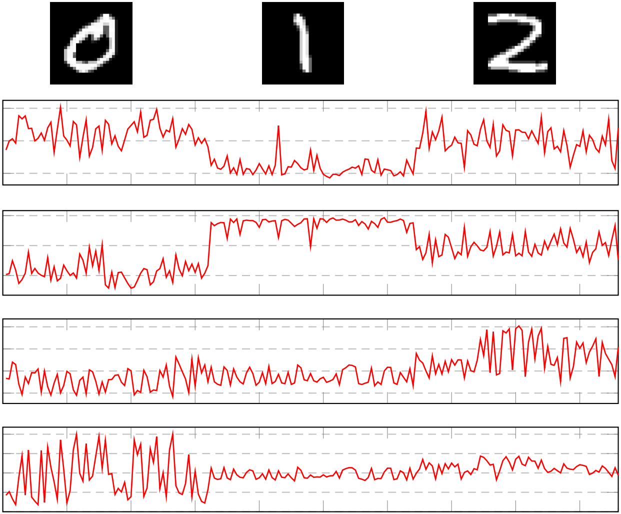

Figure: Leading four eigenvectors of $D^{-\frac{1}{2}}KD^{-\frac{1}{2}}$ for MNIST data (**red**), versus Gaussian equivalent model (**black**), and theoretical findings (**blue**).

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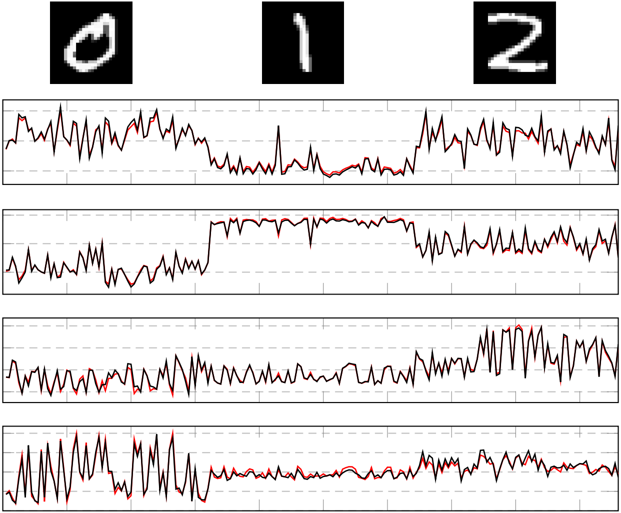

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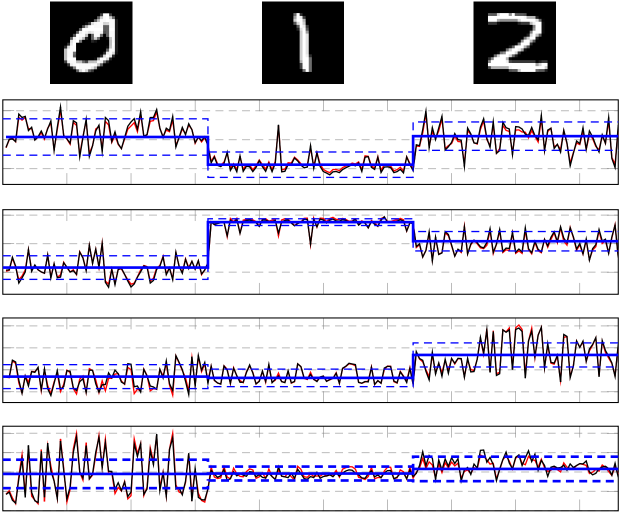

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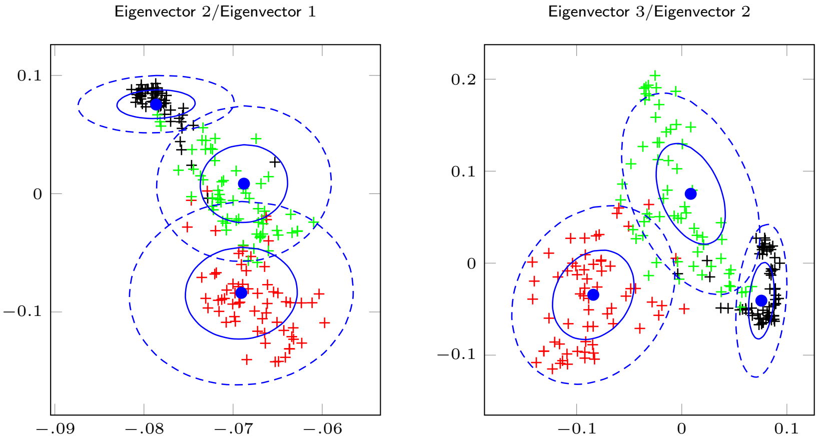

Figure: 2D representation of eigenvectors of L, for the MNIST dataset. Theoretical means and 1- and 2-standard deviations in **blue**. Class 1 in **red**, Class 2 in **black**, Class 3 in **green**.

Outline

Other Results and Perspectives

Objectives:

- **Kernel methods.**
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- **Signal processing on graphs, further graph inference, etc.**
  - 💡 Making graph methods random.

Thank you.