Application of Random Matrix Theory to Future Wireless Networks

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Outline

1. Cognitive Radios
2. Exploration: Spectrum Hole Detection
3. Exploration: User Detection and Power Inference
4. Exploitation: Optimal Ergodic Rate
5. Perspectives and Conclusion
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General system model

Primary Network

Secondary Network
Cognitive Radio Setting:
- The secondary network is entitled to reuse efficiently spectrum holes,
  - so to minimally interfere the primary network;
  - so to maximise the throughput of secondary transmissions;
  - with little or no feedback from the primary network, i.e. autonomously.
Cognitive Radio Setting:

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- To this end, the secondary (or smart, cognitive) network must
  - learn about the environment: this is the exploration phase.
  - communicate within the secondary network: this is the exploitation phase.
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- Both scenarios are inherently multi-dimensional:
  - efficient exploration requires large sensor networks;
  - networks may be composed of multiple users with possibly multiple antennas.
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We will discuss both exploration and exploitation phases as (possibly large) multivariate systems.
1  Cognitive Radios

2  Exploration: Spectrum Hole Detection

3  Exploration: User Detection and Power Inference

4  Exploitation: Optimal Ergodic Rate

5  Perspectives and Conclusion
We assume the scenario of a cognitive radio made of

- a primary network of $K$ transmitters, equipped with $n_1, \ldots, n_K$ antennas;
- a secondary network in sensing mode, equipped of $N$ collocated sensors.

Depending on prior information, the secondary network will try to infer the presence of a signal (we will assume a unique transmitter) and try to infer the transmit powers of the $K$ transmitters. This information allows for:

- the detection of spectrum holes;
- the evaluation of the optimal secondary coverage;
- ideally, the elaboration of a 'map' of space-frequency resources.

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At time $m$, the secondary network receives $y^{(m)} \in \mathbb{C}^N$ as

$$y^{(m)} = \sum_{k=1}^{K} \sqrt{P_k} H_k x_k^{(m)} + \sigma w^{(m)}$$
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Formulation of the exploration problem

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This information allows for:
  - the detection of spectrum holes;
  - the evaluation of the optimal secondary coverage;
  - *ideally*, the elaboration of a ‘map’ of space-frequency resources.
We consider the model

\[ y^{(m)} = \begin{cases} 
\sigma w^{(m)}, & (H_0) \\
\sqrt{P} H x^{(m)} + \sigma w^{(m)}, & (H_1)
\end{cases} \]
We consider the model

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We wish to confront the hypotheses \( H_0 \) and \( H_1 \) given the data matrix \( Y \triangleq [y^{(1)}, \ldots, y^{(M)}] \in \mathbb{C}^{N \times M} \).
Problem formulation

- We consider the model

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- We consider, in a Bayesian framework, the Neyman-Pearson test ratio

\[ C(Y) \triangleq \frac{P_{H_1|Y,I(Y)}}{P_{H_0|Y,I(Y)}} \]

with prior information \( I \) on \( H, x^{(m)}, \sigma, \ldots \).
We assume prior statistical and deterministic knowledge $I$ on $H, \sigma, P$. 

In the following, we derive the case $P = 1$, $\sigma$ known and the knowledge about $H$ conveys unitary invariance $E[tr HH^H]$ known: this is what we assume here; $E[tr Q]$ unknown but such that $E[tr Q]$ is known; rank($HH^H$) known.
We assume prior statistical and deterministic knowledge $I$ on $H, \sigma, P$

Using the **maximum entropy principle** (MaxEnt), a prior $P_{(H,\sigma,P)}(H, \sigma, P)$ can be derived

$$P_{Y|\mathcal{H}_i,I}(Y) = \int_{(H,\sigma,P)} P_{Y|\mathcal{H}_i,I,H,\sigma,P}(Y)P_{(H,\sigma,P)}(H, \sigma, P)d(H, \sigma, P)$$
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- we derive the case $P = 1$, $\sigma$ known and the knowledge about $H$ conveys unitary invariance
  - $E[\text{tr } HH^H]$ known: this is what we assume here;
  - $E[HH^H] = Q$ unknown but such that $E[\text{tr } Q]$ is known;
  - $\text{rank}(HH^H)$ known.

- we compare alternative methods when $P = 1$ and $\sigma$ are unknown.
by MaxEnt, $X$, $W$ are standard Gaussian matrix with $X_{ij}, W_{ij} \sim \mathcal{CN}(0, 1)$.
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**Under $\mathcal{H}_0$:**
- $Y = \sigma W$

$$P_{Y|\mathcal{H}_0, I}(Y) = \frac{1}{(\pi \sigma^2)^{NM}} e^{-\frac{1}{\sigma^2} \text{tr} YY^H}.\]
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\]

**Under \(\mathcal{H}_1\):**

\[
Y = \begin{bmatrix} \sqrt{P} H & \sigma I_N \end{bmatrix} \begin{bmatrix} X \\ W \end{bmatrix}
\]

\[
P_{Y|\mathcal{H}_1}(Y) = \int_{\Sigma \geq 0} P_{Y|\Sigma, \mathcal{H}_1}(Y, \Sigma) P_{\Sigma}(\Sigma) d\Sigma
\]

with \(\Sigma = \mathbb{E}[y^{(1)}(1)y^{(1)H}] = HH^H + \sigma^2 I_N\).
Evaluation of $P_{Y|\mathcal{H}_1}(Y)$

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From unitary invariance of $H$, denoting $\Sigma = UGU^H$, diag$(G) = (g_1, \ldots, g_n, \sigma^2, \ldots, \sigma^2)$

$$P_{Y|\mathcal{H}_1}(Y) = \int_{U(N) \times (\sigma^2, \infty)^n} P_{Y|UGU^H, \mathcal{H}_1}(Y, U, G) P_U(U) P_{(g_1, \ldots, g_n)}(g_1, \ldots, g_n) dU dg_1 \ldots dg_n$$

where
Exploration: Spectrum Hole Detection

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**Under $H_0$:**
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  - if $H$ is Gaussian, $P_{(g_1-\sigma^2, \ldots, g_n-\sigma^2)}$ is the joint eigenvalue distribution of a central Wishart;
Theorem (Neyman-Pearson test)

The ratio $C(Y)$ when the receiver knows $n = 1$, $P = 1$, $E[\frac{1}{N} \text{tr} HH^H] = 1$ and $\sigma^2$, reads

$$C(Y) = \frac{1}{N} \sum_{l=1}^{N} \frac{\sigma^2(N+M-1)e^{\sigma^2 + \frac{\lambda_l}{\sigma^2}}}{\prod_{i=1, i \neq l}^{N}(\lambda_i - \lambda_l)} J_{N-M-1}(\sigma^2, \lambda_l)$$

with $\lambda_1, \ldots, \lambda_N$ the eigenvalues of $YY^H$ and where

$$J_k(x, y) \triangleq \int_x^{+\infty} t^k e^{-t - \frac{y}{t}} dt.$$
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- contrary to energy detector, $\sum_i \lambda_i$ is not a sufficient statistic;
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- non trivial dependency on $\lambda_1, \ldots, \lambda_N$
- contrary to energy detector, $\sum_i \lambda_i$ is not a sufficient statistic;
- integration over $\sigma^2$ (or $P$ when $P \neq 1$) is difficult.
Figure: ROC curve for single-source detection, $K = 1$, $N = 4$, $M = 8$, $\text{SNR} = -3$ dB, FAR range of practical interest, with signal power $E = 0$ dBm, either known or unknown at the receiver.
Comparison to energy detector

Figure: ROC curve for single-source detection, $K = 1$, $N = 4$, $M = 8$, $\text{SNR} = -3 \text{ dB}$, FAR range of practical interest, with signal power $E = 0 \text{ dBm}$, either known or unknown at the receiver.
Exploration: Spectrum Hole Detection

Unknown power and noise variances

- Bayesian approaches:

\[ P_{Y|H_i,l}(Y) = \int_{\mathbb{R}_+^2} P_{Y|H_i,\sigma,P}(Y) P(\sigma,P)(\sigma,P)d(\sigma,P) \]

- limited by computational complexity (two-dimension numerical integration);
- inconsistence in MaxEnt uninformative priors on \( \sigma, P \).
Bayesian approaches:

\[
P_{\mathcal{Y}|\mathcal{H}_i}(\mathcal{Y}) = \int_{\mathbb{R}_+^2} P_{\mathcal{Y}|\mathcal{H}_i,\sigma}(\mathcal{Y}) P_{(\sigma,P)}(\sigma,P) d(\sigma,P)
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non-parametric GLRT approach:

\[
C_{\text{GLRT}}(\mathcal{Y}) \triangleq \frac{\sup_{\mathcal{H},\sigma} P_{\mathcal{Y}|\mathcal{H}_1}(\mathcal{Y})}{\sup_{\sigma} P_{\mathcal{Y}|\mathcal{H}_0}(\mathcal{Y})}
\]

- \(C_{\text{GLRT}}(\mathcal{Y})\) expresses as a monotonic function of \(\frac{\max_{i} \lambda_i}{\frac{1}{N} \sum_{i} \lambda_i}\);
- excludes prior information on \(H, \sigma, P\).
Bayesian approaches:

\[ P_{Y|\mathcal{H}_i,i}(Y) = \int_{\mathbb{R}_+^2} P_{Y|\mathcal{H}_i,\sigma,P}(Y) P(\sigma,P)(\sigma,P) d(\sigma,P) \]

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non-parametric GLRT approach:

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\( C_{\text{GLRT}}(Y) \) expresses as a monotonic function of \( \frac{\max_i \lambda_i}{\frac{1}{N} \sum_i \lambda_i} \);
• excludes prior information on \( H, \sigma, P \).

ad-hoc methods, such as conditioning number:

\[ C_{\text{cond}}(Y) \triangleq \frac{\max_i \lambda_i}{\min_i \lambda_i} \]

• based on large random matrix considerations: under \( \mathcal{H}_0 \), as \( N/M \to c \)

\[ \frac{\max_i \lambda_i}{\min_i \lambda_i} \overset{a.s.}{\to} \frac{(1 + \sqrt{c})^2}{(1 - \sqrt{c})^2} \]

• totally empirical.
Figure: ROC curve for a priori unknown \(\sigma^2\) of the Neyman-Pearson test, conditioning number method and GLRT, \(K = 1, N = 4, M = 8, \text{SNR} = 0\) dB. For the Neyman-Pearson test, both uniform and Jeffreys prior, with exponent \(\beta = 1\), are provided.
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and wish to infer \( P_1, \ldots, P_K \).
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\[
Y = \sum_{k=1}^{K} \sqrt{P_k} H_k X_k + \sigma W = \begin{bmatrix} \sqrt{P_1} H_1 & \cdots & \sqrt{P_K} H_K \end{bmatrix} \begin{bmatrix} X_1 \\ \vdots \\ X_K \end{bmatrix} + \sigma W
\]
\( \triangleq \begin{bmatrix} H P_1^{\frac{1}{2}} \\ \vdots \\ H P_K^{\frac{1}{2}} \end{bmatrix} X \)
Problem Statement

- We now consider the model

$$y^{(m)} = \sum_{k=1}^{K} \sqrt{P_k} H_k x_k^{(m)} + \sigma w^{(m)}$$

and wish to infer $P_1, \ldots, P_K$.
- With $Y = [y^{(1)}, \ldots, y^{(M)}]$, this can be rewritten

$$Y = \sum_{k=1}^{K} \sqrt{P_k} H_k X_k + \sigma W = \left[ \sqrt{P_1} H_1 \cdots \sqrt{P_K} H_K \right] \cdot \underbrace{\begin{bmatrix} X_1 \\ \vdots \\ X_K \end{bmatrix}}_{\equiv \sqrt{HP}} + \sigma W = \begin{bmatrix} HP_{\frac{1}{2}} & \sigma I_N \end{bmatrix} \begin{bmatrix} X \\ W \end{bmatrix}.$$
Problem Statement

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\[
y^{(m)} = \sum_{k=1}^{K} \sqrt{P_k} H_k x^{(m)}_k + \sigma w^{(m)}
\]

and wish to infer \(P_1, \ldots, P_K\).

- With \(Y = [y^{(1)}, \ldots, y^{(M)}]\), this can be rewritten

\[
Y = \sum_{k=1}^{K} \sqrt{P_k} H_k X_k + \sigma W = \begin{bmatrix} \sqrt{P_1} H_1 \\ \vdots \\ \sqrt{P_K} H_K \end{bmatrix} \begin{bmatrix} X_1 \\ \vdots \\ X_K \end{bmatrix} + \sigma W = \begin{bmatrix} HP^{1/2} \\ \sigma I_N \end{bmatrix} \begin{bmatrix} X \\ W \end{bmatrix}.
\]

- If \(H, (X^T W^T)\) are unitarily invariant, \(Y\) is unitarily invariant.
We now consider the model

$$y^{(m)} = \sum_{k=1}^{K} \sqrt{P_k} H_k x_k^{(m)} + \sigma w^{(m)}$$

and wish to infer $P_1, \ldots, P_K$.

With $Y = [y^{(1)}, \ldots, y^{(M)}]$, this can be rewritten

$$Y = \sum_{k=1}^{K} \sqrt{P_k} H_k X_k + \sigma W = \left[ \sqrt{P_1} H_1 \cdots \sqrt{P_K} H_K \right] \left[ \begin{array}{c} X_1 \\ \vdots \\ X_K \end{array} \right] + \sigma W = \left[ H P^{1/2} \right] X + \sigma W = \left[ H P^{1/2} \right] X + \sigma I_N \left[ \begin{array}{c} X \\ W \end{array} \right].$$

If $H$, $(X^T W^T)$ are unitarily invariant, $Y$ is unitarily invariant.

Most information about $P_1, \ldots, P_K$ is contained in the eigenvalues of $B_N \triangleq \frac{1}{M} YY^H$. 
The classical approach requires to evaluate $P_{P_1, \ldots, P_K|Y}$

- assuming Gaussian parameters, this is similar to previous calculus
- leads to a sum of two-dimensional integrals
- prohibitively expensive to evaluate even for small $N$, $n_k$, $M$
Assuming dimensions $N, n_k, M$ grow large, large dimensional random matrix theory provides

- a link between:
  - the “observation”: the limiting spectral distribution (l.s.d.) of $B_N$;
  - the “hidden parameters”: the powers $P_1, \ldots, P_K$, i.e. the l.s.d. of $P$.

- consistent estimators of the hidden parameters.
**Definition.** The Stieltjes transform $m_F(z)$, of a distribution function $F$ is defined as

$$m_F(z) = \int \frac{1}{\omega - z} dF(\omega).$$

Knowing the Stieltjes transform of $F$ is equivalent to knowing $F$ (similar to Fourier transform).
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For simplicity, consider the sample covariance matrix model

$$Y \triangleq T^{1/2}X \in \mathbb{C}^{N \times n}, \quad B_N = \frac{1}{n}YY^H \in \mathbb{C}^{N \times N}, \quad B_N = \frac{1}{n}Y^H Y \in \mathbb{C}^{n \times n}$$

where $T \in \mathbb{C}^{N \times N}$ has eigenvalues $t_1, \ldots, t_K$, $t_k$ with multiplicity $N_k$ and $X \in \mathbb{C}^{N \times n}$ is i.i.d. zero mean, variance 1.
**Definition.** The Stieltjes transform \( m_F(z) \), of a distribution function \( F \) is defined as

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For simplicity, consider the *sample covariance matrix* model

\[
Y \overset{\Delta}{=} T^{\frac{1}{2}} X \in \mathbb{C}^{N \times n}, \quad B_N = \frac{1}{n} Y Y^H \in \mathbb{C}^{N \times N}, \quad B_N = \frac{1}{n} Y^H Y \in \mathbb{C}^{n \times n}
\]

where \( T \in \mathbb{C}^{N \times N} \) has eigenvalues \( t_1, \ldots, t_K, t_k \) with multiplicity \( N_k \) and \( X \in \mathbb{C}^{N \times n} \) is i.i.d. zero mean, variance 1.

If \( F^T \Rightarrow T \), then \( m_{F^B_N}(z) = m_{B_N}(z) \xrightarrow{a.s.} m_F(z) \) such that

\[
m_T \left( -1/m_F(z) \right) = -z m_F(z) m_F(z)
\]

with \( m_F(z) = c m_F(z) + (c - 1) \frac{1}{z} \) and \( N/n \to c \).
Complex integration

- From Cauchy integral formula, denoting $C_k$ a contour enclosing only $t_k$,

$$t_k = \frac{1}{2\pi i} \oint_{C_k} \frac{\omega}{\omega - t_k} d\omega$$
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After the variable change $\omega = -1/m_F(z)$,

$$t_k = \frac{N}{N_k} \frac{1}{2\pi i} \oint_{C_{E,k}} z m_F(z) \frac{m'_F(z)}{m^2_F(z)} dz,$$
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When the system dimensions are large,

$$m_F(z) \simeq m_{B_N}(z) \overset{\Delta}{=} \frac{1}{N} \sum_{k=1}^{N} \frac{1}{\lambda_k - z}, \quad \text{with} \quad (\lambda_1, \ldots, \lambda_N) = \text{eig}(B_N) = \text{eig}(YY^H).$$
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- Dominated convergence arguments then show

$$t_k - \hat{t}_k \xrightarrow{a.s.} 0 \quad \text{with} \quad \hat{t}_k = \frac{N}{N_k} \frac{1}{2\pi i} \oint_{C_{E,k}} z m_{B_N}(z) \frac{m'_{B_N}(z)}{m^2_{B_N}(z)} dz.$$
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- When the system dimensions are large,
  \[ m_F(z) \sim m_{B_N}(z) \Delta \frac{1}{N} \sum_{k=1}^{N} \frac{1}{\lambda_k - z}, \quad \text{with} \quad (\lambda_1, \ldots, \lambda_N) = \text{eig}(B_N) = \text{eig}(YY^H). \]

- Dominated convergence arguments then show
  \[ t_k - \hat{t}_k \overset{\text{a.s.}}{\to} 0 \quad \text{with} \quad \hat{t}_k = \frac{N}{N_k} \frac{1}{2\pi i} \oint_{C_{E,k}} z m_{B_N}(z) \frac{m'_B(z)}{m_B^2(z)} dz = \frac{n}{N_k} \sum_{m \in \mathcal{N}_k} (\lambda_m - \mu_m) \]

  with $\mathcal{N}_k$ the indexes of cluster $k$ and $\mu_1 < \ldots < \mu_N$ are the ordered eigenvalues of the matrix $\text{diag}(\lambda) - \frac{1}{n} \sqrt{\lambda} \sqrt{\lambda}^T$, $\lambda = (\lambda_1, \ldots, \lambda_N)^T$. 
Extending \( \mathbf{Y} \) with zeros, our model is a "double sample covariance matrix"

\[
\mathbf{Y} = \mathbf{H} \mathbf{P}^{1/2} \mathbf{\Sigma} \mathbf{I}_N \mathbf{0} \mathbf{0} \mathbf{X} \mathbf{W}.
\]

Limiting distribution of \( \frac{1}{M} \mathbf{Y} \mathbf{Y}^H \)

**Theorem (l.s.d. of \( \mathbf{B}_N \))**

Let \( \mathbf{B}_N = \frac{1}{M} \mathbf{Y} \mathbf{Y}^H \) with eigenvalues \( \lambda_1, \ldots, \lambda_N \). Denote \( m_{\mathbf{B}_N}(z) \equiv \frac{1}{M} \sum_{k=1}^{M} \frac{1}{\lambda_k - z} \), with \( \lambda_i = 0 \) for \( i > N \). Then, for \( M/N \to c \), \( N/n_k \to c_k \), \( N/n \to c_0 \), for any \( z \in \mathbb{C}^+ \),

\[
m_{\mathbf{B}_N}(z) \xrightarrow{a.s.} m_{\mathcal{F}}(z)
\]

with \( m_{\mathcal{F}}(z) \) the unique solution in \( \mathbb{C}^+ \) of

\[
\frac{1}{m_{\mathcal{F}}(z)} = -\sigma^2 + \frac{1}{f(z)} \left[ \frac{c_0 - 1}{c_0} + m_P \left( -\frac{1}{f(z)} \right) \right], \quad \text{with} \quad f(z) = (c - 1)m_{\mathcal{F}}(z) - czm_{\mathcal{F}}(z)^2.
\]
Theorem (Estimator of $P_1, \ldots, P_K$)

Let $\mathbf{B}_N \in \mathbb{C}^{N \times N}$ be defined as in Theorem 2, and $\lambda = (\lambda_1, \ldots, \lambda_N)$, $\lambda_1 < \ldots < \lambda_N$. Assume that asymptotic cluster separability condition is fulfilled for some $k$. Then, as $N$, $n$, $M \to \infty$,

$$\hat{P}_k - P_k \overset{a.s.}{\to} 0,$$

where

$$\hat{P}_k = \frac{NM}{n_k(M-N)} \sum_{i \in \mathcal{N}_k} (\eta_i - \mu_i)$$

with $\mathcal{N}_k$ the set indexing the eigenvalues in cluster $k$ of $F$, $\eta_1 < \ldots < \eta_N$ the eigenvalues of $\text{diag}(\lambda) - \frac{1}{N} \sqrt{\lambda} \sqrt{\lambda}^T$ and $\mu_1 < \ldots < \mu_N$ the eigenvalues of $\text{diag}(\lambda) - \frac{1}{M} \sqrt{\lambda} \sqrt{\lambda}^T$. 

solution is computationally simple, explicit, and the final formula compact.
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cluster separability condition is fundamental. This requires

- for all other parameters fixed, the $P_k$ cannot be too close top one another: source separation problem.
- for all other parameters fixed, $\sigma^2$ must be kept low: low SNR undecidability problem.
- for all other parameters fixed, $M/N$ cannot be too low: sample deficiency issue (not such an issue though).
- for all other parameters fixed, $N/n$ cannot be too low: diversity issue.

exact spectrum separability is an essential ingredient (known for very few models to this day).
Figure: Histogram of the cluster-mean approach and of $\hat{P}_k$ for $k \in \{1, 2, 3\}$, $P_1 = 1/16$, $P_2 = 1/4$, $P_3 = 1$, $n_1 = n_2 = n_3 = 4$ antennas per user, $N = 24$ sensors, $M = 128$ samples and $\text{SNR} = 20$ dB.
Figure: Normalized mean square error of largest estimated power $\hat{P}_3, P_1 = 1/16$, $P_2 = 1/4$, $P_3 = 1$, $n_1 = n_2 = n_3 = 4$, $N = 24$, $M = 128$. Comparison between classical, moment and Stieltjes transform approaches.
1 Cognitive Radios

2 Exploration: Spectrum Hole Detection

3 Exploration: User Detection and Power Inference

4 Exploitation: Optimal Ergodic Rate

5 Perspectives and Conclusion
We assume, from the exploration phase, that power $Q_f$ can be transmitted in bandwidth $B_f$. 
Problem statement

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- The exploitation phase consists in optimally using these power resources.
- We consider the uplink scenario of
  - an $N$-antenna base station;
  - $K$ users equipped with $n_1, \ldots, n_K$ antennas;
  - Kronecker channels at all pairs $H_{k,f} = R_{k,f}^\frac{1}{2} X_{k,f} T_{k,f}^\frac{1}{2} \in \mathbb{C}^{N \times nk}$ at frequency $B_f$;
  - colored noise with covariance $\Sigma$. 

![Diagram of an $N$-antenna base station with $K$ users equipped with antennas $n_1, \ldots, n_K$, illustrating the Kronecker channels $H_{k,f}$ and colored noise $\Sigma$.]
Due to mobility, we wish to optimize the uplink ergodic sum rate (per antenna),

\[
\mathcal{I}(\mathbf{P}_1, \ldots, \mathbf{P}_K, \mathbf{F}) \triangleq \frac{1}{N} \sum_{f=1}^{F} \frac{|B_f|}{\sum_{f'} |B_{f'}|} \mathbb{E} \left[ \log \det \left( \mathbf{I}_N + \sum_{k=1}^{K} \Sigma_f^{-\frac{1}{2}} \mathbf{H}_{k,f} \mathbf{P}_{k,f} \mathbf{H}_{k,f}^H \Sigma_f^{-\frac{1}{2}} \right) \right].
\]
Due to mobility, we wish to optimize the uplink ergodic sum rate (per antenna),

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\mathcal{I}(P_{1,1}, \ldots, P_{K,F}) \triangleq \frac{1}{N} \sum_{f=1}^{F} \frac{|B_f|}{\sum_{f'} |B_{f'}|} \mathbb{E} \left[ \log \det \left( \mathbf{I}_N + \sum_{k=1}^{K} \Sigma_f^{-\frac{1}{2}} \mathbf{H}_{k,f} P_{k,f} \mathbf{H}_{k,f}^H \Sigma_f^{-\frac{1}{2}} \right) \right].
\]

Determine the sum rate maximizing precoders \( P_{k,f}^* \)

\[
(P_{1,1}^*, \ldots, P_{K,F}^*) = \arg \max_{\{P_{k,f}\}} \mathcal{I}(P_{1,1}, \ldots, P_{K,F}).
\]

\[
\sum_{k=1}^{K} \text{tr} P_{k,f} \leq Q_f
\]
Due to mobility, we wish to optimize the \textit{uplink ergodic sum rate} (per antenna),

\[ I(P_1,1, \ldots, P_K,F) \triangleq \frac{1}{N} \sum_{f=1}^{F} \frac{|B_f|}{\sum_{f'} |B_{f'}|} \mathbb{E} \left[ \log \det \left( I_N + \sum_{k=1}^{K} \Sigma_f^{-1} H_{k,f} P_{k,f} H_{k,f}^H \Sigma_f^{-1} \right) \right]. \]

Determine the \textit{sum rate maximizing precoders} \( P_{k,f}^* \)

\[ (P_{1,1}^*, \ldots, P_{K,F}^*) = \arg \max_{\{P_{k,f}\}} I(P_1,1, \ldots, P_K,F). \]

\[ \sum_{k=1}^{K} \text{tr} P_{k,f} \leq Q_f \]

\textbf{Simplifying assumptions:}
- Problem can be treated for each \( f \) independently. We then assume \( F = 1 \).
- Taking \( \Sigma = \sigma^2 I_N \) does not restrict generality.
The stochastic character of $H_{k,f}$ makes things difficult.

We instead find a deterministic approximation $I^\circ(P_1, \ldots, P_K)$ for $I(P_1, \ldots, P_K)$ such that

$$I^\circ(P_1, \ldots, P_K) - I(P_1, \ldots, P_K) \to 0$$

as $N, n_1, \ldots, n_K \to \infty$, and denote

$$(P_1^\circ, \ldots, P_K^\circ) = \arg \max_{\{P_k\}} I^\circ(P_1, \ldots, P_K).$$
The stochastic character of $H_{k,f}$ makes things difficult.

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as $N, n_1, \ldots, n_K \to \infty$, and denote

$$(P_1^\circ, \ldots, P_K^\circ) = \arg \max \mathcal{I}^\circ(P_1, \ldots, P_K).$$

We can show

$$\mathcal{I}(P_1^*, \ldots, P_K^*) - \mathcal{I}(P_1^\circ, \ldots, P_K^\circ) \to 0.$$
With $B_N \triangleq \sum_{k=1}^{K} H_k P_k H_k^H \left( H_k = R_k^{1/2} X_k T_k^{1/2} \right)$, notice that

\[
\mathcal{I}(P_1, \ldots, P_K) = E \left[ \frac{1}{N} \log \det \left( I_N + \frac{1}{\sigma^2} B_N \right) \right]
= E \left[ \int_{\sigma^2}^{\infty} \left( \frac{1}{\omega} - \frac{1}{N} \text{tr} \left( B_N + \omega I_N \right)^{-1} \right) d\omega \right]
= E \left[ \int_{\sigma^2}^{\infty} \left( \frac{1}{\omega} - m_{B_N}(-\omega) \right) d\omega \right].
\]
With $B_N \triangleq \sum_{k=1}^{K} H_k P_k H_k^H$ ($H_k = R_k^{1/2} X_k T_k^{1/2}$), notice that

$$I(P_1, \ldots, P_K) = E \left[ \frac{1}{N} \log \det \left( I_N + \frac{1}{\sigma^2} B_N \right) \right]$$

$$= E \left[ \int_{\sigma^2}^{\infty} \left( \frac{1}{\omega} - \frac{1}{N} \text{tr} (B_N + \omega I_N)^{-1} \right) d\omega \right]$$

$$= E \left[ \int_{\sigma^2}^{\infty} \left( \frac{1}{\omega} - m_{B_N}(-\omega) \right) d\omega \right].$$

It suffices to find a deterministic equivalent for $m_{B_N}(z)$. 


**Theorem (Deterministic equivalent of the Stieltjes transform)**

*Under some mild conditions on the $R_k$ and $T_k$ matrices, as $N, n_1, \ldots, n_K \to \infty$*

$$m_{B_N}(z) - m_N(z) \xrightarrow{a.s.} 0$$

where

$$m_N(z) = \frac{1}{N} \text{tr} \left( -z \left[ \sum_{k=1}^{K} \bar{e}_k(z)R_k + I_N \right] \right)^{-1}$$

and $\{\bar{e}_i(z)\}, i \in \{1, \ldots, K\}$, form the unique solution to

$$e_i(z) = \frac{1}{n_i} \text{tr} R_i \left( -z \left[ \sum_{k=1}^{K} \bar{e}_k(z)R_k + I_N \right] \right)^{-1}$$

$$\bar{e}_i(z) = \frac{1}{n_i} \text{tr} T_i^{\frac{1}{2}} P_i T_i^{\frac{1}{2}} \left( -z \left[ e_i(z)T_i^{\frac{1}{2}} P_i T_i^{\frac{1}{2}} + I_{n_i} \right] \right)^{-1}$$

**Theorem (Deterministic equivalents of the sum rate)**

*Under similar conditions, with* \( I(P_1, \ldots, P_K) = \frac{1}{N} \mathbb{E} \log \det \left( I_N + \frac{1}{\sigma^2} B_N \right) \) *and* \( z = -\sigma^2 \),

\[
I(P_1, \ldots, P_K) - I^\circ(P_1, \ldots, P_K) \to 0
\]

*where*

\[
I^\circ(P_1, \ldots, P_K) = \frac{1}{N} \log \left| I_N + \sum_{k=1}^{K} \bar{e}_k R_k \right| + \sum_{k=1}^{K} \frac{1}{N} \log \left| I_{n_k} + e_k \tilde{T}_k^{-\frac{1}{2}} P_k \tilde{T}_k^{\frac{1}{2}} \right| - \sigma^2 \sum_{k=1}^{K} \frac{n_k}{N} \bar{e}_k e_k.
\]
Theorem (Deterministic equivalents of the sum rate)

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\]

- It remains to maximize \( I^\circ(P_1, \ldots, P_K) \) over \( P_1, \ldots, P_K \) with \( \sum_k \text{tr} P_k \leq Q \).
Theorem (Deterministic equivalents of the sum rate)

Under similar conditions, with
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and \( z = -\sigma^2 \),

\[ I(P_1, \ldots, P_K) - I^\circ(P_1, \ldots, P_K) \rightarrow 0 \]

where

\[ I^\circ(P_1, \ldots, P_K) = \frac{1}{N} \log \left| I_N + \sum_{k=1}^{K} \tilde{e}_k R_k \right| + \sum_{k=1}^{K} \frac{1}{N} \log \left| I_{n_k} + e_k T_k^{1/2} P_k T_k^{1/2} \right| - \sigma^2 \sum_{k=1}^{K} \frac{n_k}{N} \tilde{e}_k e_k. \]

- It remains to maximize \( I^\circ(P_1, \ldots, P_K) \) over \( P_1, \ldots, P_K \) with \( \sum_k \text{tr} P_k \leq Q \).
- This is obtained by iterative waterfilling
  - \( P_k \) and \( T_k \) have the same eigenspaces
  - with \( \text{eig}(P_k^\circ) = (p_{k,1}, \ldots, p_{k,n_k}) \)

\[ p_{k,i} = \left( \mu - \frac{1}{e_k^{\circ} t_{k,i}} \right)^+, \quad e_k^\circ = e_k(P_1^\circ, \ldots, P_K^\circ), \quad \mu \text{ set to satisfy } \sum_k \text{tr} P_k = Q. \]
Simulation results

Figure: Ergodic MAC sum rate for an $N = 4$ antenna receiver and $K = 4$ single-antenna transmitters under sum power constraint. Every user transmit signal has different correlation patterns at the receiver, and different path losses. Deterministic equivalents (det. eq.) against simulation (sim.), with uniform (uni.) or optimal (opt.) power allocation.
Outline

1. Cognitive Radios
2. Exploration: Spectrum Hole Detection
3. Exploration: User Detection and Power Inference
4. Exploitation: Optimal Ergodic Rate
5. Perspectives and Conclusion
The road ahead

- **signal sensing:**
  - optimal hypothesis tests require symmetry, are often computationally prohibitive;
  - situations with more side information demand simpler tests, based on eigen-structure;
  - more realistic scenarios with cooperation will demand a further improvement of such tests.
The road ahead

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- optimal hypothesis tests require *symmetry*, are often *computationally prohibitive*;
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**statistical inference:**
- many methods have been proposed recently (moments, direct inversion, Stieltjes transform . . . )
  - Stieltjes transform approach seems the most powerful;
  - Stieltjes transform approach *suffers when separability is lost*;
  - estimating number of sources remains.
- test performance must be better evaluated;
- there is *room for extension* to more realistic/involved models.
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- capacity evaluation in multi-dimensional networks is progressing fast;
- optimal feedback and cooperation needs to be developed with similar random matrix tools.
Perspectives and Conclusion

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Future cognitive radio communications with  
- open-access communications,  
- secondary networks coordination,  
will demand  
- a characterization of **what information to share**,  
- a merger between random matrix theory and game theory,  
- a stronger effort on **graph-oriented random matrix theory**.
Publications in Journals: (7 accepted / 2 submitted)


Perspectives and Conclusion

List of Publications (2)

**Publications in International Conferences: (16 accepted / 3 submitted)**

- **Patents: (4 owned by ST-Ericsson)**
  - R. Couillet, M. Debbah, *Application no. 08368028.0* “Process and apparatus for performing initial carrier frequency offset in an OFDM communication system”
  - R. Couillet, S. Wagner, *Application no. 09368025.4* “Precoding process for a transmitter of a MU-MIMO communication system”
  - R. Couillet, *Application no. 09368030.4* “Process for estimating the channel in an OFDM communication system, and receiver for doing the same”
Books and book chapters: (1 book / 2 book chapters)

Random Matrix Methods for Wireless Communications [book]
Theoretical random matrix tools (finite dimensional analysis, limiting spectral laws, free probability, deterministic equivalents, statistical inference) and applications to wireless communications (SU-MIMO, MU-MIMO, CDMA, detection, estimation, channel modelling).

- Authors: R. Couillet and M. Debbah
- Publisher: Cambridge University Press
- Year: 2011 (to appear)

Chapter “Random matrix theory” on reminders of random matrix theory and especially statistical inference methods.

- Chapter authors: R. Couillet and M. Debbah
- Publisher: CRC Press, Taylor & and Francis Group
- Year: 2011 (to appear)

Orthogonal Frequency Division Multiple Access Fundamentals and Applications [book chapter 13]
Chapter “Fundamentals of OFDMA Synchronization” on theoretical considerations and application tools for time offset and frequency offset regulation in OFDM and OFDMA systems.

- Chapter authors: R. Couillet and M. Debbah
- Publisher: Auerbach Publications, CRC Press, Taylor & and Francis Group
- Year: 2010
Thank you for your attention.